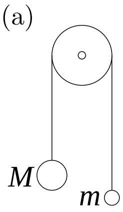
1B Dynamics
Solutions 2001

(a)



T.1 Pulleys. (a) Let the heights be z_1 and z_2. There is only one degree of freedom, because the fixed length of the string constrains $z_1 = -z_2 + c$, where c is a constant related to the choice of origin. We'll assume $c = 0$ from now on.

The energies are

$$T = \frac{M}{2}\dot{z}_1{}^2 + \frac{m}{2}\dot{z}_2{}^2 \tag{1}$$

and

$$V = Mgz_1 + mgz_2. \tag{2}$$

We eliminate z_2 and choose to use z_1 to describe our one degree of freedom. The energy is

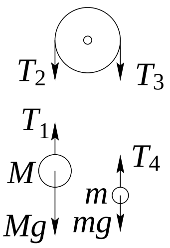
$$E = T + V = \frac{M+m}{2}\dot{z}_1{}^2 + (M-m)gz_1. \tag{3}$$

By the energy method ($\frac{dE}{dt} = 0$),

$$(M+m)\ddot{z}_1\dot{z}_1 + (M-m)g\dot{z}_1 = 0 \tag{4}$$

$$\Rightarrow \ddot{z}_1 = -\frac{(M-m)}{(M+m)}g. \tag{5}$$

Sanity check: If $M > m$, the larger mass falls; if $M = m$ then $\ddot{z}_1 = 0$.
End of energy method. □



Using forces, we have to work out the tension in the string. Because the string is massless, it requires zero force to accelerate it, so T_1 and T_2 are equal, and T_3 and T_4 are equal. And because the pulley is massless, it requires no torque to accelerate it, so T_2 and T_3 are equal. So in part (a) there is just one tension $T = T_1 = T_2 = T_3 = T_4$. The upward acceleration of mass M is given by

$$M\ddot{z}_1 = T_1 - Mg. \tag{6}$$

The upward acceleration of mass m is given by

$$m\ddot{z}_2 = T_4 - mg. \tag{7}$$

Eliminating $T_1 = T_4$ by subtracting (7) from (6), and using $\ddot{z}_2 = -\ddot{z}_1$,

$$(M+m)\ddot{z}_1 = -(M-m)g. \tag{8}$$

variable	dimension
m	M
M	M
g	LT^{-2}
$\ddot{z}_1$	LT^{-2}
4	2

This leads to the answer (5).

Notice that when using forces, we have to introduce the tension T, then use simultaneous equations to eliminate it.

Dimensional analysis can also be applied to this problem. There are four variables and two dimensional constraints, so we expect two dimensionless groups. They can be chosen to be m/M and $\ddot{z}_1/g$, from which we deduce

$$\ddot{z}_1/g = f(m/M), \tag{9}$$

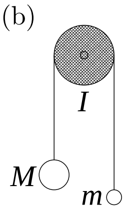
where f is a dimensionless function, or, equivalently,

$$\ddot{z}_1 = f(m/M)g. \tag{10}$$

The function f cannot be found by dimensional analysis. In fact, from the solution (5),

$$f(x) = -\frac{1-x}{1+x}. \tag{11}$$

(b)



(b) By the energy method: The angular velocity ω of the pulley is related to the velocity of the string $\dot{z}_1$ by $\dot{z}_1 = a\omega$, since there is no slip. The energy is:

$$E = \frac{1}{2}(M+m)\dot{z}_1^2 + \frac{1}{2}\frac{I}{a^2}\dot{z}_1^2 + (M-m)gz_1. \tag{12}$$

Differentiating and setting to zero,

$$\ddot{z}_1 = -\frac{(M-m)}{\left(M+m+\frac{I}{a^2}\right)}g. \tag{13}$$

End of energy method. □

variable	dimension
m	M
M	M
g	LT^{-2}
$\ddot{z}_1$	LT^{-2}
I	ML^2
a	L
6	3

By forces, this problem would take more work. We have to introduce two tensions, T_2 and T_3, which are not equal, then eliminate them both. The details of this longer approach are omitted.

Dimensional analysis is also interesting. There are three dimensionless groups:

$$\left(\frac{m}{M}\right), \left(\frac{\ddot{z}_1}{g}\right), \text{ and } \left(\frac{I}{Ma^2}\right) \tag{14}$$

are one possible choice. Dimensional analysis thus tells us that

$$\ddot{z}_1 = g \times f\left[\left(\frac{m}{M}\right), \left(\frac{I}{Ma^2}\right)\right], \tag{15}$$

where $f(x, y)$ is a dimensionless function [which in fact is

$$f(x,y) = -\frac{1-x}{1+x+y}\Bigg]. \tag{16}$$

T.2 Spring 1. The total energy is

$$E = \frac{1}{2}m\dot{z}^2 + mgz + \frac{1}{2}kz^2, \tag{17}$$

where z is the compression (if positive) or extension (if negative) of the spring, relative to its unstretched length. By the energy method,

$$m\ddot{z}\dot{z} + mg\dot{z} + kz\dot{z} = 0 \tag{18}$$

so the equation of motion is

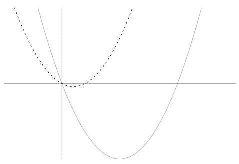
$$\ddot{z} = -g - \frac{k}{m}z. \tag{19}$$

The vertical acceleration is zero when the right hand side is zero, *i.e.*, when $z = -mg/k$. So the mass can sit in equilibrium if it's at a height such that the force from the spring (magnitude kz) is equal and opposite to the weight (magnitude mg). The solution of the equation of motion is sinusoidal oscillation about this equilibrium point:

$$z(t) = -mg/k + A\sin(\omega t + \phi), \tag{20}$$

where $\omega^2 = k/m$, and A and ϕ are determined by the initial conditions. [Skipped steps: introduce the displacement from equilibrium, $x = z - (-mg/k)$, and find the equation of motion for x, which is $\ddot{x} = -(k/m)x$.] *Notice the unstretched length l has not appeared in the solution.*

On the moon, the value of g is smaller. This changes the equilibrium point [(mg/k) is smaller, so the mass does not hang so low]. But the change in g has no effect on the frequency $\omega = \sqrt{k/m}$. One way of thinking about this is that changing g changes the linear term mgz in the potential energy



Sketch of the potential energy on earth (solid line) and on the moon (dashed line).

$$V(z) = mgz + \frac{1}{2}kz^2, \tag{21}$$

but it has no effect on the quadratic term, and it is always quadratic terms in potentials that determine oscillation frequencies, since ('equation zero')

$$\omega^2 = \frac{\partial^2 V}{\partial z^2} \Big/ m, \tag{22}$$

and when you differentiate twice, what you obtain is the coefficient of the quadratic term. Linear terms make no difference to second derivatives.

variable	dimension
m	M
g	LT^{-2}
k	MT^{-2}
l	L
ω	T^{-1}
5	3

Using dimensional analysis, the answer depends on whether we include the unstretched length, which could, in principle, have some relationship to the period of small oscillations – indeed, if we use the spring and mass as a simple pendulum, then this length *will* appear in the expression for the period. We want to find how ω depends on the other variables; we need to find $5 - 3 =$ two dimensionless groups. One group is $(\omega^2 m/k)$. Another is (mg/kl), which is the ratio of the weight of the mass to the force exerted by the spring when we double its length. From these two groups we can deduce

$$\left(\frac{\omega^2 m}{k}\right) = F\left(\frac{mg}{kl}\right) \tag{23}$$

so that the dependence of ω must have the form

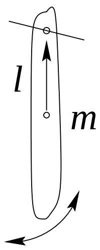
$$\omega = \left(\frac{k}{m}\right)^{1/2} F\left(\frac{mg}{kl}\right), \tag{24}$$

where F is a dimensionless function. This answer would leave open the possibility that the frequency *does* depend on the strength of gravity. However, if we further assume that there is no dependence on the unstretched length l (and you could argue for that by a thought experiment in which you replace the spring by another with identical k and different l), then dimensional analysis tells us that

$$\omega = \kappa \left(\frac{k}{m}\right)^{1/2} \tag{25}$$

where κ is a dimensionless constant that is *independent of the strength of gravity*.

Pretty neat, hey? Purely on dimensional grounds, you can tell that the vertical oscillations have the same frequency on the moon and on the earth.



T.3 Compound pendulum. It can be useful to define the *radius of gyration* k by $I_0 = mk^2$. This is the radius of a simple cylindrical body with identical moment of inertia.

The moment of inertia about the axis is $I = I_0 + ml^2 = m(k^2 + l^2)$, and the total energy is

$$E = T + V = \frac{1}{2} I \dot{\theta}^2 + mgl(1 - \cos\theta), \tag{26}$$

so, by the energy method:

$$m(l^2 + k^2)\ddot{\theta} = -mgl \sin\theta. \tag{27}$$

For small θ, we approximate $\sin\theta \simeq \theta$ and get

$$\ddot{\theta} = -\frac{gl}{(l^2 + k^2)}\theta. \tag{28}$$

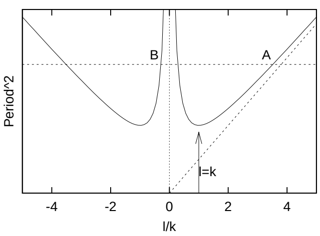
The solution of this equation is simple harmonic motion with frequency

$$\omega = \sqrt{\frac{gl}{k^2 + l^2}}. \tag{29}$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}. \tag{30}$$

Interpretation: At small l, ($k^2 \gg l^2$) rotational inertia dominates, and the restoring couple (which scales as gl) becomes small because the centre of mass rises little when the pendulum is displaced; so for small l the period becomes large. At large l, the pendulum becomes like a simple pendulum, with period increasing with l. Thus for both large and small l, the period increases, and there is a minimum period at intermediate l. In order to sketch a graph of the period, we differentiate $(k^2 + l^2)/gl$ with respect to l and find that it is zero at $l = k$.



We replicate the graph of $T(l)$ for negative l using $T(l) = T(|l|)$ [the meaning of negative l is that we are suspending the pendulum from a point on the other side of the centre of mass; when we do this, small oscillations of the pendulum happen about the equilibrium position $\theta = \pi$ instead of $\theta = 0$]. The figure shows a sketch of T^2 versus l/k. Notice that for any chosen achievable period, there are four suspension points that give the same period (except for the special case where $l = k$).

T.4 Safety rope. If we put two springs of constant k end to end, the resulting spring has spring constant equal to $k/2$, since for a given force its extension is twice as great. Thus a length l of stretchable rope has spring

variable	dimension
f	MLT^{-2}
m	M
g	LT^{-2}
k^*	MLT^{-2}
l	L
5	3

constant *inversely proportional* to l. We can write $k = k^*/l$, where k^* is a property of the rope, namely the force that, when applied to *any* piece of the rope, doubles its length.

Using dimensional analysis, there are two dimensionless groups to be found: we can choose (f/mg) and (mg/k^*). Thus we obtain

$$f \; = \; mg \; G\!\left(\frac{mg}{k^*}\right), \tag{31}$$

where G is a dimensionless function. Notice that this implies that f has *no dependence on l*. That's a pretty strong result!

The motion can be solved using energy conservation. The rope stretches until an extension x such that

$$\frac{k^* x^2}{2l} = m(l+x)g. \tag{32}$$

$(l+x)$ is the total length of the extended rope. This is a quadratic equation for x, in general. Rather than writing out the general solution $x = -b \pm \sqrt{\ldots}$, let's focus on the special cases. In the case of a *stiff* rope, the maximum extension x will be *small* compared to the length l, so we can replace $(l+x)$ by l in (32). We then find that the maximum force is

$$F = kx = \sqrt{2k^* mg}. \tag{33}$$

If the rope is very *stretchable*, then we expect x to be much *greater* than l, so we can replace $(l+x)$ by x. This gives

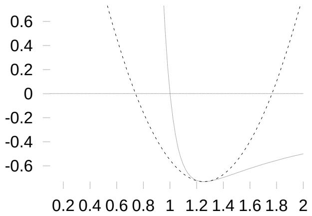
$$F = 2mg. \tag{34}$$

These two answers both agree with the fact we found by dimensional analysis, that the maximum force is independent of the length l. An interesting thought for bungie-jumpers.

T.5 Oscillation.

The equation of motion (found by the energy method, for example) is

$$m\ddot{x} = -\frac{\partial V}{\partial x} = -\frac{A}{x^2} + \frac{12B}{x^{13}}. \tag{35}$$

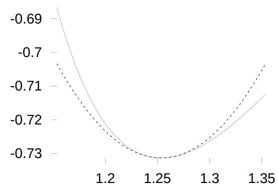


The acceleration is zero at the $x = x_0$ such that

$$\frac{A}{x^2} = \frac{12B}{x^{13}}, \tag{36}$$

i.e.,

$$x_0^{11} = \frac{12B}{A}. \tag{37}$$



We now Taylor-expand V about this equilibrium point:

$$V \simeq V(x_0) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} (x-x_0)^2 \ldots \tag{38}$$

so the equation of motion for small deviations $x - x_0$ from equilibrium is

$$m\ddot{x} = -\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}(x - x_0) \tag{39}$$

which implies, if the second derivative is positive, simple harmonic motion with frequency

$$\omega^2 = \frac{\partial^2 V}{\partial x^2}\bigg/ m. \qquad \text{(‘Equation 0’)} \tag{40}$$

Now, we can evaluate the second derivative by brute force:

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} = +2\frac{A}{x_0^3} - \frac{13 \times 12B}{x_0^{14}}, \tag{41}$$

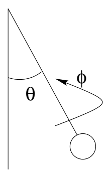
or we can use hygienic differentiation™:

$$\begin{aligned}
\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} &= \left.\frac{\partial}{\partial x}\left(\frac{1}{x^{13}}\right)\left(Ax^{11} - 12B\right)\right|_{x=x_0} && (42)\\
&= \left.\left(\frac{\partial}{\partial x}\frac{1}{x^{13}}\right)\left(Ax^{11} - 12B\right)\right|_{x=x_0} + \left.\frac{1}{x^{13}}\left(\frac{\partial}{\partial x}Ax^{11}\right)\right|_{x=x_0} && (43)\\
&= 0 + \frac{1}{x_0^{13}}\left(11Ax_0^{10}\right) && (44)\\
&= \frac{11A}{x_0^3} && (45)
\end{aligned}$$

The oscillation frequency is thus given by

$$\omega^2 = \frac{1}{m}\frac{11A}{x_0^3} = \frac{11A}{m}\left(\frac{A}{12B}\right)^{3/11} \tag{46}$$

At this stage, it would be good to check dimensions. A has dimensions of energy times length, *i.e.*, ML^3T^{-2}. A/B has dimensions L^{-11}, so the right hand side has dimensions $L^3T^{-2}L^{-3} = T^{-2}$. Incidentally, the dependence of ω on A and B could have been deduced by dimensional analysis, except for the dimensionless constant.

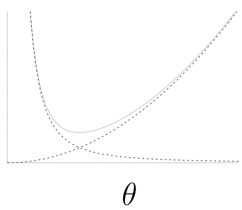


T.6 Conical pendulum. The energies are:

$$T = \frac{1}{2}m(l\dot\theta)^2 + \frac{1}{2}m(l\dot\phi\sin\theta)^2, \qquad V(\theta) = mgl(1 - \cos\theta). \tag{47}$$

The angular momentum J about the vertical axis (*i.e.*, the z-component of $\mathbf{J}$, the angular momentum about the suspension point on the axis) is

$$J = \frac{1}{2}m(l\sin\theta)^2\dot\phi, \tag{48}$$



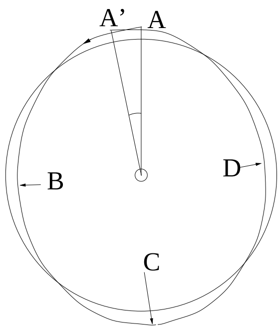
Sketch of $V(\theta)$ and $\frac{1}{2}\frac{J^2}{ml^2\sin^2\theta}$ (dashed lines) and the effective potential energy (solid line).

and we can use $J =$ constant to eliminate $\dot\phi$. [The z-component of $\mathbf{J}$ is constant because the two forces acting, the tension and the weight, have respectively zero couple and a couple with no $z-$component.] We rearrange $E = T + V$ into the form

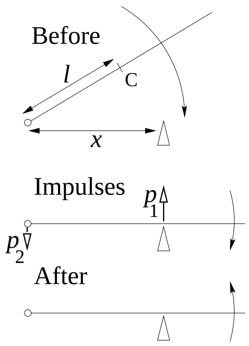
$$E = V_{\text{eff}}(\theta) + \frac{1}{2}ml^2\dot\theta^2, \tag{49}$$

where
$$V_{\text{eff}}(\theta) = V(\theta) + \frac{1}{2}\frac{J^2}{ml^2\sin^2\theta}. \tag{50}$$
(a) If the mass moves steadily in a circle then θ is a constant, so we are stationary in the effective potential. Therefore $\theta = \theta_0$ is such that
$$\left.\frac{\partial V_{\text{eff}}}{\partial \theta}\right|_{\theta_0} = mgl\sin\theta - \frac{J^2\cos\theta}{ml^2\sin^3\theta} = 0; \tag{51}$$
Putting the angular momentum about the center, $J = (ml^2\sin^2\theta)\dot{\phi}$, we find that the angular velocity $\Omega = \dot{\phi}$ satisfies
$$\Omega^2 = \frac{g}{l\cos\theta_0}. \tag{52}$$
(b) Oscillation about θ_0 is related to the 2nd derivative $\left.\frac{\partial^2 V_{\text{eff}}}{\partial\theta^2}\right|_{\theta_0}$ of $V_{\text{eff}}(\theta)$ about θ_0 [Recall 'equation zero']. The oscillation frequency ω is given by:
$$\omega^2 = \frac{1}{ml^2}\left.\frac{\partial^2 V}{\partial\theta^2}\right|_{\theta_0} \tag{53}$$
[If it's not clear where this came from, apply the energy method to the energy, expanded as a Taylor series:
$$E = V_{\text{eff}}(\theta_0) + \frac{1}{2}\left.\frac{\partial^2 V_{\text{eff}}}{\partial\theta^2}\right|_{\theta_0}(\Delta\theta)^2 + \ldots + \frac{1}{2}ml^2\dot{\theta}^2.\ ] \tag{54}$$
The second derivative is
$$\frac{\partial^2 V}{\partial\theta^2} = mgl\cos\theta + \frac{J^2}{ml^2}\frac{2\cos^2\theta + 1}{\sin^4\theta}. \tag{55}$$
Simplifying, we get
$$\omega^2 = \frac{1}{ml^2}\left.\frac{\partial^2 V}{\partial\theta^2}\right|_{\theta_0} = \frac{g}{l\cos\theta}(1 + 3\cos^2\theta). \tag{56}$$
When $\theta_0 \approx 0$, $\cos\theta \approx 1$ so $\omega^2 \approx 4\Omega^2$ and $\omega \approx 2\Omega$, so the radial wobble occurs at roughly twice the frequency of rotation. The circular motion is thus perturbed into an ellipse centred on the axis of rotation. One way of making this result obvious is to notice that for small θ_0, the spherical surface on which the mass moves is locally a parabola of the form $h(x, y) = \frac{k}{2}(x^2 + y^2)$, and for small θ_0, the kinetic energy is $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. The x and y motions are not coupled to each other. Each undergoes independent simple harmonic motion. The two simple harmonic motions happen to have the same frequency so the resulting path is an ellipse.

If we now take into account the first correction terms in small θ_0, the ratio of ω to 2Ω, the frequency for perfect ellipses, is given by
$$\frac{\omega^2}{4\Omega^2} = 1 - \frac{3\theta_0^2}{8} + \cdots, \tag{57}$$
i.e.,
$$\frac{\omega}{2\Omega} = 1 - \frac{3\theta_0^2}{16} + \cdots, \tag{58}$$



so the small oscillation will not quite have finished in one big revolution. The orientation of the ellipse will precess in the direction of $\dot{\theta}$. [See the figure, which shows an orbit starting from a maximum of θ at A, going through a minimum at B, a second maximum at C, a second minimum at D, and arriving 'late' at A.] The orientation of the ellipse will change through 90 degrees in about $\frac{(8/3)}{\theta_0^2}$ revolutions. For example if θ_0 is 0.5 (about 30 degrees), the orbit will precess through 90 degrees in about 10 revolutions. You can check this prediction with a clamped piece of string.



T.7 Door stop. I'll assume it is obvious that the door stop should be placed *vertically* at half the height of the door, and focus on the interesting *radial* part.

Method: during the collision, the doorstop and the hinges both deliver impulses to the door to change its motion from rotating one way about the hinge to rotating the other way. Find the location x for the doorstop such that the impulse p_2 at the hinge is zero.

Issue: it's unclear at the outset whether the answer will depend on whether the collision is elastic. We can handle both cases (elastic and inelastic) by having the angular velocities ω_{before} and ω_{after} be different from each other, and solve for the impulses for that general case.

Details: Change in angular momentum about hinge:

$$I_{\text{hinge}}\,(\omega_{\text{after}} - \omega_{\text{before}}) = p_1 x, \tag{59}$$

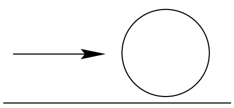
where $I_{\text{hinge}} = I_0 + ml^2 = \frac{4}{3}ml^2$. [Parallel axes theorem.] Change in linear momentum:

$$ml\,(\omega_{\text{after}} - \omega_{\text{before}}) = p_1 - p_2. \tag{60}$$

If we set $p_2 = 0$ then we can divide (59) by (60):

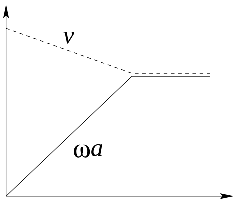
$$\frac{4}{3}l = x. \tag{61}$$

Answer: Put the doorstop half way up the height of the door, and two-thirds of the way from the hinge to the edge of the door. [Remember the door width is $2l$.]



T.8 Snooker. Qualitative description: If hit along the centre, the ball would immediately-post-impulse have linear momentum, but no angular momentum about its centre of mass; it would therefore be *slipping*. The friction decelerates the linear motion, and exerts a couple about the centre of mass, causing the ball to start rotating. As it rotates faster and moves more slowly, a point will come when the linear velocity and angular velocity are compatible, so the ball starts to roll without slipping.

Now, assuming the standard model of friction, the frictional force that opposes the sliding is a constant (F), independent of the relative velocity. (The friction force is proportional to the perpendicular force, which is not varying in this problem.) The non-slipping condition is $v = \omega a$, so when sketching v and ω on a single graph, it makes sense to multiply ω by a, so that non-slipping corresponds to the two graphs meeting.



$$m\dot{v} = -F \quad \longrightarrow \quad v = v_0 - \frac{F}{m}t \tag{62}$$

$$I\dot{\omega} = Fa \quad \longrightarrow \quad a\omega = 0 + \frac{Fa^2}{I}t = \frac{5F}{2m}t \tag{63}$$

After $t_0 = \frac{2mv_0}{7F}$, $v = \omega a$, at which point it starts rolling.

To make it roll right away, the momentum impulse Δp should be such that it sets up compatible linear motion and rotation. We can find ω and v from the impulsive couple and Newton's second law respectively; if the impulse is delivered horizontally at height h above the centre of mass,

$$h\Delta p = I\omega = \frac{2}{5}ma^2\omega \tag{64}$$

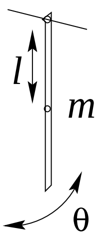
$$\Delta p = mv. \tag{65}$$

Using $v = a\omega$ we obtain

$$h = \frac{2a}{5}, \tag{66}$$

so you should hit the ball $\frac{7a}{5}$ above table, that is, at 70% of the full height of the ball.

Lagrangian and Hamiltonian dynamics

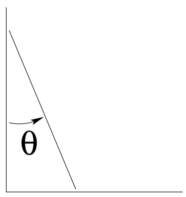


T.9 Ladder. (a) The potential energy is

$$V = mgl(1 - \cos\theta). \tag{67}$$

The kinetic energy can be written as the sum of the rotational kinetic energy plus the translational energy of the centre of mass:

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 \tag{68}$$

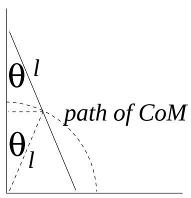


So the Lagrangian is

$$L = T - V = \frac{1}{2}(I + ml^2)\dot{\theta}^2 - mgl(1 - \cos\theta). \tag{69}$$

The equation of motion is

$$(I + ml^2)\ddot{\theta} = -mgl\sin\theta. \tag{70}$$



(b) The ladder has length $2l$. The potential energy is $V = -mgl(1 - \cos\theta)$. The kinetic energy can be written as the sum of the rotational kinetic energy plus the translational energy of the centre of mass: $T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. Now, as the sketch shows (using two similar triangles), the centre of mass moves along a circular path centred on the origin, and $(\dot{x}^2+\dot{y}^2)=l^2\dot{\theta}^2$, So the Lagrangian is

$$L = T - V = \frac{1}{2}(I+ml^2)\dot{\theta}^2 + mgl(1-\cos\theta). \tag{71}$$

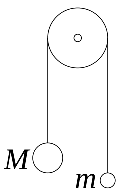
Notice that this Lagrangian is *identical* to that of the compound pendulum, except that the potential energy term has flipped sign. Thus the falling ladder is *equivalent* to an upside-down compound pendulum.

The conjugate momentum is

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = (I+ml^2)\dot{\theta}. \tag{72}$$

The equation of motion is

$$(I+ml^2)\ddot{\theta} = +mgl\sin\theta. \tag{73}$$



T.10 Pulley Galore. (a) Pulley by Lagrangian methods:

$$L = T - V = \frac{M+m}{2}\dot{z_1}^2 - (M-m)gz_1. \tag{74}$$

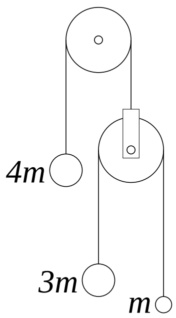
Conjugate momentum:

$$\frac{\partial L}{\partial\dot{z}_1} = T - V = M + m\dot{z}_1 \tag{75}$$

Euler–Lagrange equation:

$$\frac{d}{dt}M + m\dot{z}_1 = \frac{\partial L}{\partial z_1} = -(M-m)g \tag{76}$$

$$\ddot{z}_1 = -\frac{(M-m)}{(M+m)}g. \tag{77}$$



(b) This system has two degrees of freedom. Let's answer the final question first. The most useful extreme case to think about is where the right hand masses are replaced by $4m$ and ϵm. In this limit, both the $4m$s plummet towards the ground at g, and the little guy therefore goes up at $3g$; the tension in all the strings is negligible. Similarly, in the given case, the tension in the right hand string is less than the $2mg$ that would be needed to balance the $4mg$ weight on the left because the $3mg$ mass is quite close to a state of free fall, so it's not pulling its weight.

Let's use as our two coordinates z_4, the height of the $4m$, and z_2, the distance through which the right-hand pulley rotates. Thus the height of the $3m$ is defined to be $z_2 - z_4$, and of that of the m is $-z_2 - z_4$. The Lagrangian is

$$\begin{aligned} L &= T - V \qquad (78)\\ &= \frac{1}{2}4m\dot{z}_4^2 + \frac{1}{2}3m(\dot{z}_2-\dot{z}_4)^2 + \frac{1}{2}m(\dot{z}_2+\dot{z}_4)^2 \\ &\quad -4mgz_4 - 3mg(z_2-z_4) - mg(-z_2-z_4) \qquad (79)\\ &= 4m\dot{z}_4^2 + 2m\dot{z}_2^2 - 2m\dot{z}_4\dot{z}_2 - 2mgz_2. \qquad (80)\end{aligned}$$

The conjugate momenta are

$$
\begin{aligned}
p_4 &= 8m\dot{z}_4 - 2m\dot{z}_2 && (81)\\
p_2 &= 4m\dot{z}_2 - 2m\dot{z}_4. && (82)
\end{aligned}
$$

The Euler-Lagrange equations are

$$
\begin{aligned}
\frac{d}{dt}\left[8m\dot{z}_4 - 2m\dot{z}_2\right] &= 0 && (83)\\
\frac{d}{dt}\left[4m\dot{z}_2 - 2m\dot{z}_4\right] &= -2mg. && (84)
\end{aligned}
$$

Rearranging, we can solve for the two accelerations.

$$
\begin{aligned}
8\ddot{z}_4 - 2\ddot{z}_2 &= 0 && (85)\\
-\ddot{z}_4 + 2\ddot{z}_2 &= -g && (86)
\end{aligned}
$$

$$
\begin{aligned}
7\ddot{z}_4 = -g \quad \Rightarrow \quad & \ddot{z}_4 = -g/7 && (87)\\
& \ddot{z}_2 = -g/2 - g/14 = -4g/7. && (88)
\end{aligned}
$$

So the big guy *falls* with acceleration $g/7$, and the $3m$ *falls* at $3g/7$, and the smallest mass accelerates *upwards* at $5g/7$.

T.11 Vertical state space. $L = \frac{1}{2}m\dot{z}^2 - mgz$. $p = m\dot{z}$. $H = p\dot{z} - L = \frac{1}{2}\frac{p^2}{m} + mgz$. Hamilton's equations are:

$$\frac{d}{dt}z = \frac{p}{m}; \quad \frac{d}{dt}p = -mg. \tag{89}$$

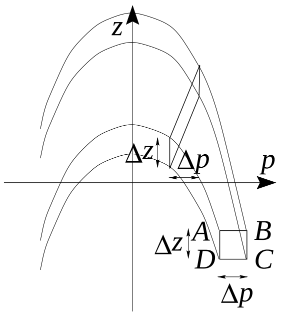
The solution for $z(t)$ and $p(t)$ is

$$p(t) = p(0) - mg \tag{90}$$

(momentum is a linear decreasing function of time);

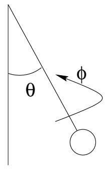
$$z(t) = z(0) + ut - \frac{1}{2}gt^2, \tag{91}$$

where $u = p(0)/m$. Since z is a parabolic function of time and p is linear with time, z is also a parabolic function of p.



From the solution for p, (90), we can see that two initial conditions that differ from each other by Δp will lead to later states that still differ by exactly Δp. From (91), we can see that initial differences in z alone will lead to equal differences in z later. And from the ut term in (91), we can see that initial differences in p will cause growing vertical differences. So the rectangle ABCD evolves into a parallelogram. But the area of the parallelogram is still $\Delta p \, \Delta z$.

T.12 Conical pendulum II.



$$L = T - V = \frac{1}{2}m(l\dot\theta)^2 + \frac{1}{2}m(l\dot\phi\sin\theta)^2 - mgl(1-\cos\theta). \tag{92}$$

Conjugate momenta:

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta \tag{93}$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = ml^2\sin^2\theta\dot\phi \tag{94}$$

Generalized forces

$$\frac{\partial L}{\partial\theta} = ml\dot\phi^2\sin\theta\cos\theta - mgl\sin\theta \tag{95}$$

$$\frac{\partial L}{\partial\phi} = 0, \tag{96}$$

so the Euler–Lagrange equations are

$$\frac{d}{dt}ml^2\dot\theta = ml\dot\phi^2\sin\theta\cos\theta - mgl\sin\theta \tag{97}$$

and

$$\frac{d}{dt}\left[ml^2\sin^2\theta\dot\phi\right] = 0,\ \ i.e.,\ \ \left[ml^2\sin^2\theta\dot\phi\right] = \text{constant}. \tag{98}$$

The Hamiltonian is

$$\begin{aligned} H(\theta,\phi,p_\theta,p_\phi) &= p_\theta\dot\theta + p_\phi\dot\phi - L & (99)\\ &= \frac{1}{2}m(l\dot\theta)^2 + \frac{1}{2}m(l\dot\phi\sin\theta)^2 + mgl(1-\cos\theta) & (100)\\ &= \frac{1}{2ml^2}p_\theta^2 + \frac{1}{2ml^2\sin^2\theta}p_\phi^2 + mgl(1-\cos\theta). & (101)\end{aligned}$$

[As usual for systems like this, H is the total energy $H = T+V$.] Hamilton's equations are

$$\begin{aligned} \dot\theta &= \frac{\partial H}{\partial p_\theta} \qquad \dot\phi = \frac{\partial H}{\partial p_\phi} & (102)\\ \dot p_\theta &= -\frac{\partial H}{\partial\theta} \qquad \dot p_\phi = -\frac{\partial H}{\partial\phi} & (103)\\ \dot\theta &= \frac{p_\theta}{ml^2} \qquad \dot\phi = \frac{p_\phi}{ml^2\sin^2\theta} & (104)\end{aligned}$$

$$\dot p_\theta = \frac{\cos\theta}{ml^2\sin^3\theta}p_\phi^2 - mgl\sin\theta\dot p_\phi = 0 \tag{105}$$

If

$$\frac{\cos\theta}{ml^2\sin^3\theta}p_\phi^2 = mgl\sin\theta \tag{106}$$

then we have a constant value of θ because $\dot p_\theta = 0$. We'll call this fixed point θ_0. Linearizing the expression for $\dot p_\theta$, the equations of motion for θ and p_θ are

Hygienic differentiation used here.

$$\dot{\theta} = \frac{p_\theta}{ml^2} \qquad \dot{p}_\theta = (\theta - \theta_0)\frac{\partial}{\partial \theta}\left[\frac{\cos\theta}{ml^2 \sin^3\theta}p_\phi^2 - mgl\sin\theta\right]_{\theta_0} \tag{107}$$

The derivative is

$$\frac{\partial}{\partial\theta}\left[\left(\frac{\cos\theta}{ml^2}p_\phi^2 - mgl\sin^4\theta\right)\frac{1}{\sin^3\theta}\right]_{\theta_0} \tag{108}$$

$$= \left[\left(\frac{-\sin\theta}{ml^2}p_\phi^2 - 4mgl\sin^3\theta\cos\theta\right)\frac{1}{\sin^3\theta}\right]_{\theta_0} \tag{109}$$

$$= \left(\frac{-\sin\theta_0}{ml^2\sin^3\theta_0}p_\phi^2 - 4mgl\cos\theta_0\right) \tag{110}$$

$$= mgl\left(\frac{-\sin^2\theta_0 - 4\cos^2\theta_0}{\cos\theta_0}\right) \tag{111}$$

$$= -mgl\left(\frac{1 + 3\cos^2\theta_0}{\cos\theta_0}\right) \tag{112}$$

Where we used (106) along the way. So θ and p_θ are related, for small $(\theta - \theta_0)$, by

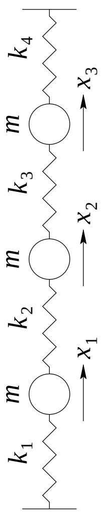
$$\dot{\theta} = \frac{p_\theta}{ml^2} \qquad \dot{p}_\theta = -mgl(\theta - \theta_0)\left(\frac{1 + 3\cos^2\theta_0}{\cos\theta_0}\right) \tag{113}$$

from which we can find that both of them perform simple harmonic motion:

$$\ddot{p}_\theta = -mgl\frac{p_\theta}{ml^2}\left(\frac{1 + 3\cos^2\theta_0}{\cos\theta_0}\right) \tag{114}$$

with frequency

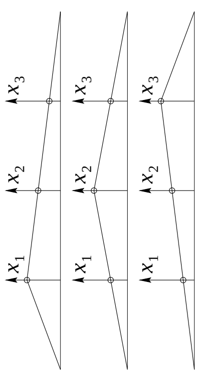
$$\frac{g}{l}\left(\frac{1 + 3\cos^2\theta_0}{\cos\theta_0}\right). \tag{115}$$



Matrices

T.13 Displaced springs. The forces are given by $\mathbf{f} = -\mathbf{Kx}$ and the potential energy is $V = \frac{1}{2}x_i K_{ij} x_j$, where

$$K_{ij} = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{pmatrix} \tag{116}$$



When a unit force is applied to $1, 2, 3$, the displacements look as shown. We can solve for the displacements by equating the applied force to the force exerted by the springs and solving for $\mathbf{x}$

$$\mathbf{f}_{\text{applied}} = \mathbf{Kx} \tag{117}$$

$$\mathbf{x} = \mathbf{K}^{-1}\mathbf{f}_{\text{applied}}. \tag{118}$$

By finding the inverse of the matrix, we can solve for all three problems at once. Alternatively, if you don't like inverting matrices, you could plug in $\mathbf{f}_{\text{applied}} = (1, 0, 0)$ and solve the simultaneous equations for $\mathbf{x}$ – in which case, you are actually inverting the matrix by hand.

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \tag{119}$$

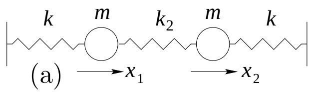
So the responses to unit forces $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are $\mathbf{K}^{-1}(1, 0, 0)^{\mathrm{T}}$, $\mathbf{K}^{-1}(0, 1, 0)^{\mathrm{T}}$, and $\mathbf{K}^{-1}(0, 0, 1)^{\mathrm{T}}$, that is, the columns of $\mathbf{K}^{-1}$,

$$\begin{array}{c} \left(\begin{array}{ccc} 3/4 & 2/4 & 1/4 \end{array} \right) \\ \left(\begin{array}{ccc} 1/2 & 1 & 1/2 \end{array} \right) \\ \left(\begin{array}{ccc} 1/4 & 2/4 & 3/4 \end{array} \right) \end{array} . \tag{120}$$

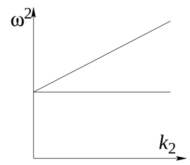
Because the matrix $\mathbf{K}^{-1}$ is *symmetric*, the displacement at j when a unit force is applied at i is equal to the displacement at i when a unit force is applied at j. For example, the displacement of 2 when the force is applied to 1 is $2/4$, and the displacement of 1 when the force is applied to 2 is $1/2$.

In passing, we note that for *any* system with a quadratic potential $V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{K}\mathbf{x}$, the displacement of degree of freedom j when a unit force is applied to degree of freedom i is equal to the the displacement at i when a unit force is applied at j. This is true because we can always choose the matrix $\mathbf{K}$ to be symmetric (it's in a quadratic form), so its inverse $\mathbf{K}^{-1}$ is symmetric too. And the element K^{-1}_{ij} is the displacement of j when a unit force is applied to degree of freedom i.

Normal modes



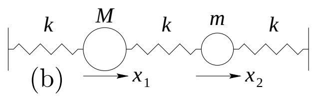
T.14 Two masses. (a) By symmetry, the modes are $\mathbf{e}^{(a)} = (1, 1)$ and $\mathbf{e}^{(b)} = (1, -1)$, whatever value the central spring constant k_2 has. The first mode does not stretch the middle spring at all, so



$$\omega_a^2 = \frac{k}{m}. \tag{121}$$

The second mode stretches the middle spring through two units, so

$$\omega_b^2 = \frac{k + 2k_2}{m}. \tag{122}$$



The modes themselves do not change with k_2.

(b) The equation of motion is $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$, where

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix} \ \& \ \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}. \tag{123}$$

The mode frequencies are given by the generalized eigenvalue equation

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0 \tag{124}$$

that is, with $\lambda = \omega^2$,

$$\begin{vmatrix} 2k - \lambda M & -k \\ -k & 2k - \lambda m \end{vmatrix} = 0 \tag{125}$$

$$Mm\lambda^2 - 2k(M+m)\lambda + 3k^2 = 0 \tag{126}$$

$$\lambda = \frac{k(M+m) \pm \sqrt{k^2(M+m)^2 - 3Mmk^2}}{Mm} \tag{127}$$

so

$$\omega^2 = \frac{k}{m}\left[1 + \frac{m}{M} \pm \sqrt{1 - \frac{m}{M} + \left(\frac{m}{M}\right)^2}\right] \tag{128}$$

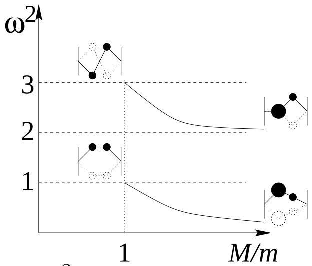
or

$$\omega^2 = \frac{k}{M}\left[1 + \frac{M}{m} \pm \sqrt{1 - \frac{M}{m} + \left(\frac{M}{m}\right)^2}\right]. \tag{129}$$

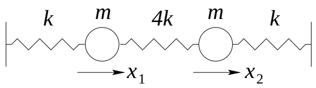
If M varies and m is constant, (128) is the more convenient expression for sketching.

When $M/m = 1$, we know from part (a) that $\omega^2/(k/m)$ is 1 or 3.

When M/m is large, m/M is tiny, and



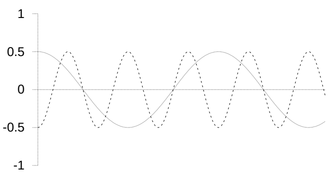
ω^2 is shown in units of k/m.



$$\omega^2/\frac{k}{m} \simeq \left[1 + \frac{m}{M} \pm \left(1 - \frac{1}{2}\frac{m}{M} + \ldots\right)\right] = \begin{cases} 2 + \frac{1}{2}\frac{m}{M} + \ldots \\ \frac{3}{2}\frac{m}{M} + \ldots \end{cases} \tag{130}$$

For large M, let's sanity-check these answers. The high frequency mode involves the tiny mass wobbling to and fro in between a genuine wall on the right and an effective wall on the left, the large mass M, which scarcely budges. The frequency of this motion should be $\omega^2 = 2k/m$ since there are two restoring springs of constant k. Good.

The low frequency mode involves the big M moving to and fro between a spring k and a second spring (k in series with k) with a negligible speck (m) attached halfway. The total spring constant is $k + k/2 = 3k/2$, and the frequency should be $\omega^2 \simeq 3k/2M$. Good.



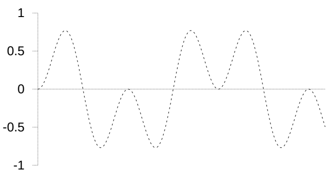
The amplitudes of the two modes.

T.15 Two mass II.

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 5k & -4k \\ -4k & 5k \end{bmatrix} \tag{131}$$

We need the eigenvalues and eigenvectors of

$$\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}. \tag{132}$$

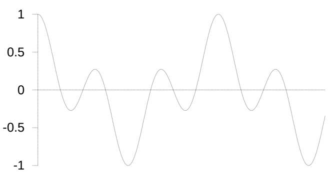


The displacement of the left mass.

We found these in the previous question. The eigenvector $(1,1)$ has eigenvalue 1 and the eigenvector $(1,-1)$ has eigenvalue 9, so the frequencies are $\omega_a = 1\omega_0$ and $\omega_b = 3\omega_0$ where $\omega_0^2 = k/m$.

(b) The general motion of the system is

$$\begin{aligned} \mathbf{x}(t) &= \sum_a A_a \cos(\omega_a t + \phi_a)\mathbf{e}^{(a)} && (133) \\ &= A_a \cos(\omega_a t + \phi_a)[1,\ 1] + A_b \cos(\omega_b t + \phi_b)[1,\ -1], && (134) \end{aligned}$$



The displacement of the right mass.

where A_a and ϕ_a control the amplitude of normal mode a. We must here satisfy the boundary conditions

$$\mathbf{x}(0) = (0,\ \ 1) \qquad \dot{\mathbf{x}}(0) = (0,\ \ 0). \tag{135}$$

By inspection, we can satisfy these constraints by setting $\phi_a = 0$ and $\phi_b = 0$ and giving the two modes equal and opposite amplitudes.

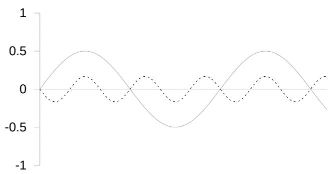
$$A_a = \frac{1}{2}, \qquad A_b = -\frac{1}{2}. \tag{136}$$

So

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{2}\left\{\cos(\omega_0 t)[1\ \ 1] - \cos(3\omega_0 t)[1\ \ -1]\right\} && (137)\\ &= \frac{1}{2}\left[(\cos(\omega_0 t) - \cos(3\omega_0 t)),\quad (\cos(\omega_0 t) + \cos(3\omega_0 t))\right]. && (138) \end{aligned}$$

These graphs can be sketched by plugging in special values of $\omega_0 t$, or by using angle formulae. such as $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

(c) When the right hand mass is hit instead of displaced, our task is to match the initial conditions

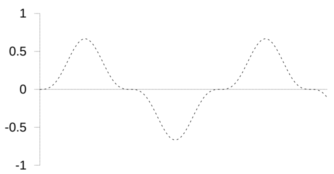


The amplitudes of the two modes.

$$\mathbf{x}(0) = (0,\ \ 0) \qquad \dot{\mathbf{x}}(0) = (0,\ \ 1). \tag{139}$$

To achieve zero displacement, we need to rotate the phase from cosine to sine.

$$\mathbf{x}(t) = A_a \sin(\omega_0 t)[1,\ \ 1] + A_b \sin(3\omega_0 t)[1,\ \ -1] \tag{140}$$



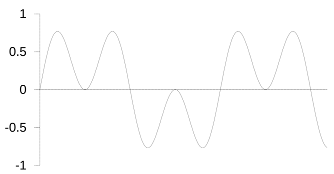
The displacement of the left mass.

What should the coefficients A_a and A_b be? We differentiate the solution (140) with respect to time, so that we can apply the second constraint:

$$\dot{\mathbf{x}}(t) = \omega_0 A_a \cos(\omega_0 t)[1,\ \ 1] + 3\omega_0 A_b \cos(3\omega_0 t)[1,\ \ -1] \tag{141}$$

So we must have

$$\begin{bmatrix} \omega_0 A_a + 3\omega_0 A_b \\ \omega_0 A_a - 3\omega_0 A_b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{142}$$



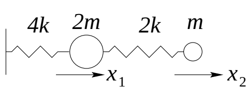
The displacement of the right mass.

Subtracting and adding, we find

$$A_a = \frac{1}{2}\frac{1}{\omega_0}, \qquad A_b = -\frac{1}{6}\frac{1}{\omega_0}. \tag{143}$$

Notice that the excitation by impulse (c), compared with excitation by plucking (b, above) produces relatively *less* of the high frequency mode.

T.16 Two mass III. The matrices are:



$$\mathbf{M} = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 6k & -2k \\ -2k & 2k \end{bmatrix} \tag{144}$$

We want the solutions of the generalized eigenvalue problem

$$\left[\mathbf{K} - \omega^2 \mathbf{M}\right]\mathbf{e} = 0. \tag{145}$$

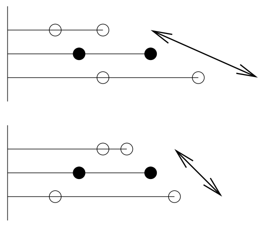
First we find the eigenvalues $\lambda = \omega^2$, omitting factors of k/m which we can put back later:

$$\begin{vmatrix} 6-2\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0 \tag{146}$$

$$\lambda^2 - 5\lambda + 4 = 0 \tag{147}$$

$$\Rightarrow \omega^2 = \begin{cases} 4(k/m) \\ 1(k/m) \end{cases} \qquad \Rightarrow \omega = \begin{cases} 2(k/m)^{1/2} \\ 1(k/m)^{1/2} \end{cases} \tag{148}$$

We solve for the eigenvectors by plugging these two values for ω^2 into (145). $\lambda = \omega^2 = 4$ gives



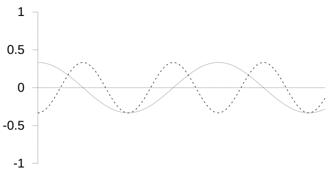
(1, 2) and (1, −1).

$$\begin{bmatrix} 6-8 & -2 \\ -2 & 2-4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \tag{149}$$

so

$$\begin{bmatrix} e_1^{(a)} \\ e_2^{(a)} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{150}$$

Similarly $\lambda = \omega^2 = 1$ gives

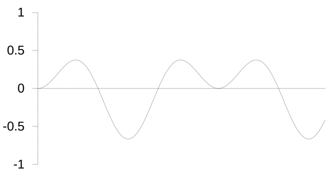


The amplitudes of the two modes.

$$\begin{bmatrix} e_1^{(b)} \\ e_2^{(b)} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{151}$$

Note that these two eigenvectors are not orthogonal. They satisfy the generalized orthogonality relationships

$$\mathbf{e}^{(a)}\mathbf{M}\mathbf{e}^{(b)} = 0, \quad \mathbf{e}^{(a)}\mathbf{K}\mathbf{e}^{(b)} = 0. \tag{152}$$

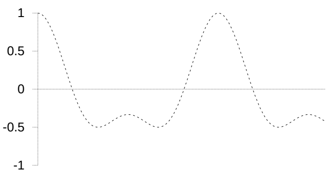


The displacement of the left mass.

When the right hand mass is displaced and both masses are stationary, we must match this initial condition $\mathbf{x}(0) = (0,\ 1)$ using

$$\mathbf{x}(t) \;=\; A_a \cos(\omega_0 t + \phi_a)[1,\ 2] + A_b \cos(2\omega_0 t + \phi_b)[1,\ -1], \tag{153}$$

where $\omega_0 = \omega_a = (k/m)^{1/2}$. We find $\phi_a = 0$ and $\phi_b = 0$ and

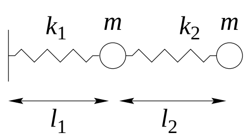


The displacement of the right mass.

$$A_a = \frac{1}{3}, \quad A_b = -\frac{1}{3}. \tag{154}$$

[If you have defined the eigenvectors differently, for example, you might have [1/2, 1] instead of [1, 2], then you will get different amplitudes for the normal modes. But the final answer for $\mathbf{x}(t)$ will be the same.]

$$\mathbf{x}(t) \;=\; \frac{1}{3}\left[\cos(\omega_0 t) - \cos(2\omega_0 t), \quad 2\cos(\omega_0 t) + \cos(2\omega_0 t)\right] \tag{155}$$



T.17 Divided spring.

This question is half back-to-front compared with traditional questions, which say 'here are **K** and **M** – tell me the modes'; this one says 'here is a mode, tell me **K**'.

The matrices are:

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{156}$$

and we are told that (1, 2) is an eigenvector. We can substitute this fact into the eigenvalue equation

$$\mathbf{K}\mathbf{e} = \omega^2 m\mathbf{e} : \tag{157}$$

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \omega^2 m \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{158}$$

$$\begin{bmatrix} k_1 - k_2 \\ k_2 \end{bmatrix} = \omega^2 m \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{159}$$

Dividing these two forces by each other, we have

$$\left[\frac{k_1 - k_2}{k_2}\right] = \frac{1}{2} \tag{160}$$

So

$$\left[\frac{k_1}{k_2}\right] = \frac{3}{2} \tag{161}$$

Now, from the safety rope question, we know that spring constants k go inversely with length, so the lengths must be in the ratio

$$\left[\frac{l_1}{l_2}\right] = \frac{2}{3}. \tag{162}$$

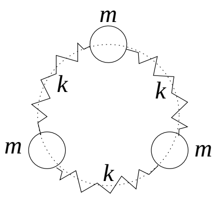
We can find the higher frequency mode without calculation by orthogonality. It must be

$$(2, -1). \tag{163}$$

The ratio of the frequencies can be found by solving for both of them. Dropping dimensional factors, since we only want a ratio,

$$\left|\mathbf{K} - \omega^2 \mathbf{1}\right| = 0 \;\Rightarrow \begin{vmatrix} 5 - \omega^2 & -2 \\ -2 & 2 - \omega^2 \end{vmatrix} = 0 \;\Rightarrow \omega^2 = 6 \text{ or } 1. \tag{164}$$

So the ratio of frequencies is $\omega_{\max}/\omega_{\min} = \sqrt{6}$.



T.18 Symmetries. Three masses moving on a circle can be solved using the same method as the four masses in a circle – see the normal modes handout. (Alternatively, you can use guessing'n'checking.) The system is symmetric under clockwise permutation of the three displacements (*i.e.*, rotation through 120 degrees), so we can find the normal modes by finding the eigenvectors of that permutation operator. The $N = 3$ eigenvectors $\mathbf{f}^{(a)}$ are given by $f_n^{(a)} = e^{i2\pi an/N}$, for $a = 0, 1, 2$. The modes are

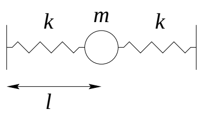
- $(1, 1, 1)$, which corresponds to steady rotation – this mode has zero frequency.
- $(1, e^{i2\pi/3}, e^{-i2\pi/3})$ and $(1, e^{-i2\pi/3}, e^{i2\pi/3})$. These modes correspond to complex travelling waves travelling clockwise and anticlockwise.

If we prefer all our modes to be real, we can take appropriate linear combinations of the two complex modes. Adding and subtracting, we obtain:

- $(2, -1, -1)$;
- $(0, 1, -1)$.

These modes are degenerate and both have frequency $\sqrt{3k/m}$.

For the eleven-mass system, the $N = 11$ eigenvectors $\mathbf{f}^{(a)}$ are given by $f_n^{(a)} = e^{i2\pi an/N}$, for $a = 0, 1, 2, \ldots, 11$.



T.19 3D spring. There are three degrees of freedom. By symmetry, the three modes must be one along the line of the springs (with frequency $2k/m$, independent of l and l_0) and two degenerate modes perpendicular to the springs. We find the frequency of the perpendicular modes by Taylor-expanding the potential. If the lateral displacement is z, then

$$V(z) = 2\frac{1}{2}ke^2 = k((z^2 + l^2)^{1/2} - l_0)^2; \tag{165}$$

the first derivative is

$$\frac{\partial V}{\partial z} = 2k((z^2 + l^2)^{1/2} - l_0)(z^2 + l^2)^{-1/2}z. \tag{166}$$

The second derivative at $z = 0$, which is what we need to find the frequency of the mode, is

$$\left.\frac{\partial^2 V}{\partial z^2}\right|_{z=0} = 2k((z^2 + l^2)^{1/2} - l_0)(z^2 + l^2)^{-1/2}\Big|_{z=0} = 2k(l - l_0)/l. \tag{167}$$

[Note that we can be hygienic when differentiating: there are many z-dependent terms in the first derivative (166), but one of them (z) is zero at $z = 0$, so we only need to differentiate that one – the others don't matter.]

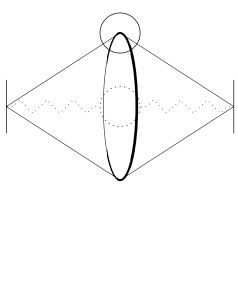
The expression we have derived is simply twice the tension $k(l - l_0)$ divided by the length l – a familiar result! So the Taylor expansion of the potential is

$$V(z) = V(0) + \frac{1}{2}(2k(l - l_0)/l)z^2 \ldots \tag{168}$$

which is like the potential of a spring with constant $k_{\text{eff}} = 2k(l - l_0)/l$; the frequency of the transverse modes is given by

$$\omega^2 = 2(k/m)(l - l_0)/l. \tag{169}$$

If the 'stretched' length l is smaller than the unstretched length l_0, then ω^2 is negative. This means the fixed point is *unstable* to the two transverse displacements. If perturbed from equilibrium, the transverse displacement grows exponentially. [To be precise, it grows initially exponentially, but once the displacement becomes large (*i.e.*, at all comparable to l), higher terms in the Taylor expansion become relevant. There will be a new stable equilibrium state, indeed, a whole circle of such states, in which the system is bent in a V shape and both springs have their unstretched lengths.]



(b) When the three springs are arranged symmetrically, there is by symmetry one transverse mode (in and out of the page), and its frequency, generalizing the two spring result, is $3(k/m)(l - l_0)/l$. As for the remaining two modes, it must be possible to find a pair that respect the three-fold symmetry of the system. The eigenvectors of the operator that rotates the in-plane displacement through 120 degrees are $(1, i)$ and $(1, -i)$ (see below for proof, if this is not familiar), and so these are normal modes of the three-spring system. They describe clockwise and anticlockwise circular motions. The two modes are degenerate, so any linear combination of them is a normal mode. Now by adding appropriate multiples of $(1, i)$ and $(1, -i)$, we can obtain any desire vector (x, y). So *any* displacement in the plane is a normal mode. If you kick the mass from the centre in any direction, it will simply oscillate in that direction. Any direction in the plane defines a normal mode.

The potential, to quadratic order, has the form $\frac{1}{2}\mathbf{x}\mathbf{K}\mathbf{x}$; and this quadratic function must be invariant under rotation of $\mathbf{x}$ through 120 degrees. The only quadratic functions having this symmetry are ones in which $\mathbf{K}$ is proportional to the identity matrix. The potential is thus $\frac{1}{2}k^*(x^2 + y^2)$, where x and y are the two in-plane displacements and k^* is the effective spring constant.

If the mass is given a kick in any direction, starting from the origin, it simply oscillates to and fro in that direction at frequency $\sqrt{k^*/m}$.

T.20 Driven system. Method: Project the state (x_1, x_2) onto the eigenvectors $(1, 1)$ and $(1, -1)$, and work out the equation of motion for the projections $u_1 = x_1 + x_2$ and $u_2 = x_1 - x_2$. The force acting on the first mass is equivalent to a force acting on each degree of freedom u_1 and u_2.

$$m\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} + \mathbf{f} \tag{170}$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = -\frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} (f/m)\sin(\omega t) \\ 0 \end{bmatrix} \tag{171}$$

$$\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = -\frac{k}{m}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} (f/m)\sin(\omega t) \\ (f/m)\sin(\omega t) \end{bmatrix} \tag{172}$$

Now, we need to recall the solution to the driven simple harmonic oscillator,

$$\ddot{u} = -\omega_0^2 u + (f/m)\sin(\omega t). \tag{173}$$

We guess a steady state solution of the form

$$u = A\sin(\omega t). \tag{174}$$

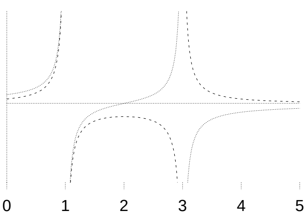
$$-A\omega^2\sin(\omega t) = -\omega_0^2 A\sin(\omega t) + (f/m)\sin(\omega t). \tag{175}$$

$$A = \frac{f/m}{(\omega_0^2 - \omega^2)}. \tag{176}$$

So the steady state solutions for u_1 and u_2 are

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{f/m}{\frac{k}{m} - \omega^2}\sin(\omega t) \\ \frac{f/m}{\frac{3k}{m} - \omega^2}\sin(\omega t) \end{bmatrix} \tag{177}$$

and x_1 and x_2 are

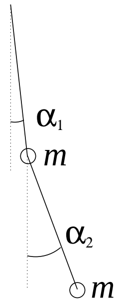


$$x_1 = \frac{1}{2}\left\{\frac{f/m}{\frac{k}{m} - \omega^2} + \frac{f/m}{\frac{3k}{m} - \omega^2}\right\}\sin(\omega t) \tag{178}$$

$$x_2 = \frac{1}{2}\left\{\frac{f/m}{\frac{k}{m} - \omega^2} - \frac{f/m}{\frac{3k}{m} - \omega^2}\right\}\sin(\omega t) \tag{179}$$

The figure shows the amplitudes of the responses of x_1 (solid line) and x_2 (dashed line) as a function of $\omega^2/(k/m)$. Notice when $\omega^2 = 2k/m$, there is

NO response in x_1, and x_2 responds in antiphase. Under these conditions, the first mass is playing the role of a stationary wall, and f is precisely the force that such a wall needs to exert for mass 2 to do its lone oscillations. At very high frequencies, mass 1 responds in antiphase and mass 2 responds in phase with the driving force.



T.21 Double pendulum. Let's first recap the dynamics of the *non-rotating* double pendulum. We assume the two masses are equal. We find the equation of motion by Lagrangian methods. Because we are interested in the motion *near the fixed point* $(\alpha_1, \alpha_2) = (0, 0)$, we will *approximate* the Lagrangian, making an approximation that is accurate for small angles.

For small angles, the masses' kinetic energy is associated almost entirely with horizontal motion; the two horizontal speeds are approximately $l\dot{\alpha}_1$ and $l\dot{\alpha}_1 + l\dot{\alpha}_2 = l(\dot{\alpha}_1 + \dot{\alpha}_2)$. So the kinetic energy is

$$T_{\text{not rotating}} \simeq \frac{1}{2}ml^2\dot{\alpha}_1^2 + \frac{1}{2}ml^2(\dot{\alpha}_1+\dot{\alpha}_2)^2 = \frac{1}{2}ml^2\left[2\dot{\alpha}_1^2 + 2\dot{\alpha}_1\dot{\alpha}_2 + \dot{\alpha}_2^2\right]. \quad (180)$$

The potential energy is

$$\begin{aligned} V &= mgl\,(1-\cos\alpha_1) + mgl\,(1-\cos\alpha_1 + 1 - \cos\alpha_2) \\ &= 2mgl\,(1-\cos\alpha_1) + mgl\,(1-\cos\alpha_2). \end{aligned} \quad (181)$$

For small angles, we can use $\cos\alpha \simeq 1 - \frac{1}{2}\alpha^2 + \ldots$ to obtain

$$V \simeq 2mgl\,\frac{\alpha_1^2}{2} + mgl\,\frac{\alpha_2^2}{2}. \quad (182)$$

Notice that both these approximated energies can be written as quadratic forms:

$$T_{\text{not rotating}} = \frac{1}{2}ml^2 \begin{bmatrix} \dot{\alpha}_1 & \dot{\alpha}_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \quad (183)$$

$$V = \frac{1}{2}mgl \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \quad (184)$$

The effect of rotation at rate ω is to add extra terms to the kinetic energy. The radial distance of the first mass from the axis is $l\alpha_1$, so the rotational kinetic energy is $\frac{1}{2}ml^2\alpha_1^2\omega^2$; for the second mass, the extra energy is $\frac{1}{2}ml^2(\alpha_1+\alpha_2)^2\omega^2$. So the total kinetic energy is

$$T = \frac{1}{2}ml^2 \begin{bmatrix} \dot{\alpha}_1 & \dot{\alpha}_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} + \frac{1}{2}ml^2\omega^2 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad (185)$$

and the Lagrangian of the rotating double pendulum is

$$\begin{aligned} L &= T - V & (186) \\ &= \frac{1}{2}ml^2 \left[\dot{\alpha}_1, \dot{\alpha}_2\right] \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} - \frac{1}{2}ml \left[\alpha_1, \alpha_2\right] \begin{bmatrix} 2g - 2\omega^2 l & -\omega^2 l \\ -\omega^2 l & g - \omega^2 l \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} & (187) \end{aligned}$$

We can think of these two quadratic forms as an effective kinetic energy T_{eff} and an effective potential V_{eff}, if we wish. We define $\mathbf{M}$ and $\mathbf{K}$ to be the two matrices in (187). We now solve for the generalized eigenvectors. In general, this would be a rather messy business, with solutions of quadratic equations running around. But in this case, the close relationship between the matrix proportional to ω^2 appearing in T and the other matrix in T means that it comes out rather nicely – the eigenvectors will be the same for all ω.

We want to find the eigenvalues, *i.e.*, the roots of

$$|\mathbf{K} - \lambda \mathbf{M}| = 0. \tag{188}$$

We divide through by ml^2 and define the natural frequency of a single pendulum by $\omega_0^2 = g/l$. [Not to be confused with ω^2; or λ, which is the square of a normal mode frequency!]

$$\left| \begin{array}{cc} 2\omega_0^2 - 2\omega^2 - 2\lambda & -\omega^2 - \lambda \\ -\omega^2 - \lambda & \omega_0^2 - \omega^2 - \lambda \end{array} \right| = 0 \tag{189}$$

Since every ω^2 is accompanied by a λ, we define $\lambda' = (\omega^2 + \lambda)/\omega_0^2$, so we can save ink and solve

$$\left| \begin{array}{cc} 2 - 2\lambda' & -\lambda' \\ -\lambda' & 1 - \lambda' \end{array} \right| = 0, \tag{190}$$

finding

$$\lambda' = 2 \pm \sqrt{2} \tag{191}$$

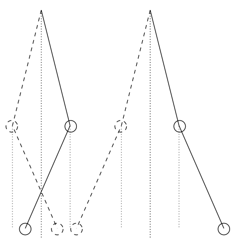
Thus the frequencies of the two normal modes are given by

$$\lambda^{1/2} = (\lambda' \omega_0^2 - \omega^2)^{1/2} = \sqrt{\left(2 \pm \sqrt{2}\right) g/l - \omega^2}. \tag{192}$$

(a) For the special case of no rotation ($\omega = 0$), these frequencies are $1.8\sqrt{g/l}$ and $0.8\sqrt{g/l}$. The corresponding displacements are given by

$$\left[\begin{array}{cc} 2 - 2\lambda' & -\lambda' \\ -\lambda' & 1 - \lambda' \end{array} \right] \left[\begin{array}{c} e_1 \\ e_2 \end{array} \right] = 0, \tag{193}$$

from which we can find the ratio of e_1 to e_2, giving



$$\left[\begin{array}{c} e_1 \\ e_2 \end{array} \right] \propto \left[\begin{array}{c} (-1 \mp \sqrt{2}) \\ (2 \pm \sqrt{2}) \end{array} \right] = \left[\begin{array}{c} -2.4 \\ 3.4 \end{array} \right] \text{ and } \left[\begin{array}{c} 0.4 \\ 0.6 \end{array} \right] \tag{194}$$

Notice that these two eigenvectors are not orthogonal. The lower angle α_2 is bigger in magnitude in both modes. You might check that they do satisfy the generalized orthogonality rule. The mode with higher frequency is shown on the left, and the lower frequency to its right.

(b) For general rotation rate ω, the eigenvector equation (193) still applies, so *the eigenvectors are the same for all* ω. The only thing that changes is the frequency (192) of each normal mode. Since both frequencies decrease with ω, there will come a critical rotation rate at which the *lower* normal mode eigenvalue, $\lambda_- = \left(2 - \sqrt{2}\right) g/l - \omega^2$, will change sign. The equation of motion for the displacement of that normal mode coordinate will therefore change from

$$\ddot{x}_a = -\omega^2 x_a \tag{195}$$

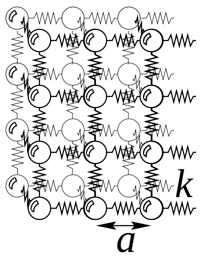
to

$$\ddot{x}_a = |\lambda_-| x_a, \tag{196}$$

whose solutions are exponentially growing and decaying functions, rather than oscillatory functions. If ω^2 exceeds the critical value, $\left(2-\sqrt{2}\right) g/l$, the fixed point changes from a stable to an unstable fixed point. For any perturbation from the fixed point, the amplitude of the component of the lower frequency normal mode will grow exponentially.

If an uncle holds a niece in the air and spins her round, there is a critical spinning rate above which the niece tends to fly round with α_1 and α_2 both large and positive.

Elasticity



T.22 Model steel.

Example of estimating k given $a = 3 \times 10^{-10}\,\mathrm{m}$: model the interatomic potential by a quadratic function with minimum at spacing a, and depth 5eV, and with curvature such that the potential is zero when the displacement is $a/2$, gives $k \simeq 1\,\mathrm{eV}/10^{-20}\,\mathrm{m}^2 = 16\,\mathrm{N/m}$.

Relate k and a to the Young's modulus, and deduce the Young's modulus of this model steel.

Consider extension e of a tiny cubical (a^3) fragment of steel containing just one bond.

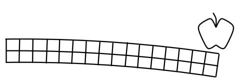
$$Y = (F/A)/(e/l) = (F/e)/a = k/a = 16/3 \times 10^{-10} = 5 \times 10^{10}\, N/m^2$$

[The true value is 2×10^{11} Pa]

We can also estimate Y_{steel} from experience. Imagine plucking a guitar string, and imagine twiddling the peg that tunes the string. From the experience of the forces required to deflect the string sideways 1 cm, or to extend the string by 1 mm, and the resulting change in pitch when this extension is imposed, we can get the information we need. Take the high E string, for example. Its diameter is thin, maybe 0.5 mm. The string is roughly 1 m long, and if we hang a jar of jam (1 kg) from it at its midpoint, it deflects by maybe 1 cm. This gives us the tension, $T = 500\,\mathrm{N}$ (from resolving forces in the long skinny triangle). Sanity check. That means that the tension is about the weight of a 50 kg child. Seems reasonable. Now, how much does extending the wire increase the tension? I'd guess that one revolution of the peg (which extends the string by, say, 1cm) would cause a major change in pitch, maybe as much as a fifth. A fifth is 3/2 in frequency, which is 9/4 in tension. So an extension of 1% is expected to double the tension from the starting value. The Young's modulus is the stress that would double the length (*i.e.*, produce a strain of 1), so it's 100 times the stress in the string, *i.e.* (with area = $(.5\,\mathrm{mm})^2$),

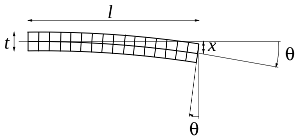
$$Y \simeq 100T/A = 100 \times 500/(25 \times 10^{-8}) = 2 \times 10^{11}\,\mathrm{N/m^2}.$$

Lucky!



Estimate the vertical deflection of an apple on a ruler. How does your answer depend on the board's thickness?

We assume that the upper half of the ruler is stretched and the lower half is compressed. From experience, I'd expect a deflection of 1 mm or 2 cm so.



Define the vertical displacement of the end to be x, the thickness, t, the length l, and the angle of the end of beam, θ. We estimate the energy stored in the ruler when it's deformed as shown. The energy is stored in the stretched and compressed parts. The energy is (typical energy density) $\times$ (volume that is deformed). The angle θ is roughly given by $\theta = x/l$, with a geometry factor of some sort. Shall we take

$$\theta = x/(2l)?$$

The maximum strain, ϵ, of the upper edge is the total extension of the upper edge, $(t/2)\theta$, over l.

$$\epsilon_{\max} = t\theta/(2l)$$

The energy density is

$$\frac{1}{2}Y\epsilon^2.$$

The strain ϵ is a linear function of distance from the midplane, so the energy density is a quadratic function of distance. Let's just use the maximum strain and multiply by half the volume of the beam (assuming that roughly half of it is at the maximum strain). The potential energy is then

$$V(x) \simeq \text{Volume } \times \text{ Energy density}$$
$$= \frac{1}{2}(tlw)\frac{1}{2}Y\epsilon^2 = \frac{1}{2}(tlw)\frac{1}{2}Y\frac{t^2(x/(2l))^2}{4l^2} = \frac{1}{2}\frac{t^3wY}{32l^3}x^2,$$

so (comparing this with a Hooke spring's $V = \frac{1}{2}kx^2$) the end of the beam behaves just like Hooke spring with constant

$$k \simeq \frac{t^3wY}{32l^3}.$$

Check dimensions:

$[F][L]^{-1} \leftrightarrow [L][F][L]^{-2}$

Notice that this scales as the *cube* of the thickness – thick planks are much harder to bend than thin ones – and it scales inverse-cubically with length, which fits with the experience that an apple deflects a long ruler much more than an equivalent short one. The linear scaling with width w makes complete sense, since two apples on two rulers, side by side, give the same deflection as one apple on one ruler.

So, let's try the apple on our model ruler.

Displacement for a 1 N apple, on a width-2 cm diving board of length 0.3 m, thickness 10^{-3} m, is predicted to be

$$1\,\mathrm{N}/k \simeq \frac{32l^3}{t^3wY} = \frac{32 \times 0.3^3}{10^{-9} \times .02 \times 2 \times 10^{11}}\,\mathrm{m} = 0.2\,\mathrm{m}.$$

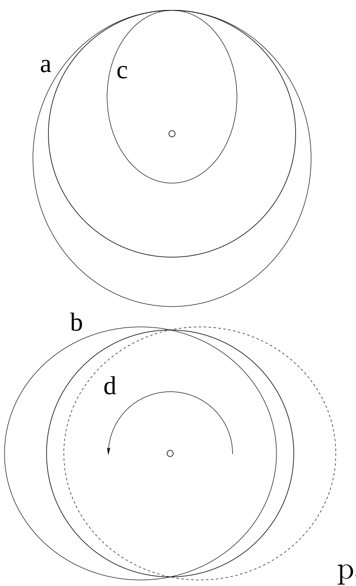
This is an embarrassingly large answer, about ten times larger than expected. The scaling with thickness t is cubic, so big errors arise from getting it a little wrong.

Our estimate of the typical strain for a given deflection is also a possible cause of error, as our estimate of k scales quadratically with the strain.

T.23 Shear strain is equivalent to compression and extension.

Orbits

T.24 Ellipses. The way to answer this sort of question is to identify all the constraints that the solution must satisfy:



1. the orbit, if it is a closed orbit in a $1/r$ potential, must be an ellipse *with the attractive origin at one focus*; [the most common error in these problems is to draw ellipses that don't satisfy this constraint.]

2. if we get onto this orbit by receiving a kick at some point P, the orbit must come back through P;

3. the tangent to the orbit at any point is in the same direction as the velocity at that point – in particular, if you know the velocity at P, you can deduce the tangent at P;

4. ellipses have various useful properties, for example,

   (a) the ellipse is symmetrical about its major axis; of course, you may not be sure initially what the direction of the major axis is, but this fact is still a useful constraint.

   (b) there's only two points on an ellipse where the tangent is perpendicular to the radius vector – they are the two points lying on the major axis.

   (c) locally, you can think of an ellipse like the parabola along which a free-falling body falls – so sometimes you can use your knowledge of free-falling bodies to figure out the local picture.

Given what we discussed in lectures, you should be able to answer all parts of this question except for the final part, 'state the changes in period of the satellite' – we didn't discuss how period is related to the orbital parameters very much. However, if you remember the precise statement of Kepler's 3rd law, you should be able to do this bit too: K3 says that $T^2 \propto a^3$, where a is the length of the semi-major axis.

OK, let's solve the problems. First, let's do b, which we did in lecture. For small radial perturbations, the orbit can be thought of as an oscillation about the original circular orbit. The frequency of the oscillation happens to be the same as the original orbital frequency, so the perturbed orbit, by massive coincidence, is a closed orbit that wobbles inside and outside the circle once per orbit.

In case b, the energy is slightly increased by the kick (the new speed, post-kick, is slightly larger), and the period is increased (because the semi-major axis is a little larger); the angular momentum is unchanged.

In case d, we obtain an ellipse that is the mirror image of b. The energy is slightly increased, just as in b, the period is increased, and the angular momentum is unchanged.

Now comes the challenge: what about case a? The kick is in the direction of the velocity, so the new velocity points in the same direction as before the kick. So the new ellipse must have a tangent in the same direction as the old circular orbit, at the kick point. The ellipse only has two points where its tangent is perpendicular to the radius vector – these are the closest and furthest distances from the attracting point, and they lie on the major axis. So the kick point, P, *must lie on the major axis of the new ellipse*. The new orbit is sketched in the figure. You can think of the motion close to P by analogy with the parabolic falling of a mass: the bigger the horizontal kick, the wider the parabola.

In case a, there is a significant increase in energy (bigger than in case b, because a kick along the direction of $\mathbf{v}$ has a much bigger effect on the magnitude of $\mathbf{v}$). The angular momentum increases and the period increases.

Case c corresponds to the other orientation of a Kepler ellipse.

T.25 Power law potentials.

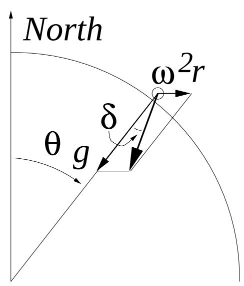
The effective potential is

$$V_{\rm eff} = \frac{1}{2}\frac{J^2}{mr^2} + V(r) \tag{197}$$

(a) If $V(r) = Ar^2/2$, the effective potential is positive and grows without limit as $r \to 0$ or $r \to \infty$, so the motion is bounded, whatever the energy. The effective potential has a single minimum and the motion is to and fro in r about this minimum. In fact the orbits are ellipses with the centre of attraction at the centre of the ellipse. This quadratic potential is identical to the potential of the conical pendulum, to leading order in the displacement from vertical.

(b) If $V(r) = A \log r/r_0$, then all motions are bounded, but large excursions away from the centre of attraction are possible, since $\log r$ is such a slowly increasing function.

(c) If $V(r) = -\frac{A}{5r^5}$, the effective potential has a maximum at $r_{\rm crit}$ and no minimum. The repulsive $1/r^2$ term is insufficient to repel the particle from $r = 0$. Possible trajectories include unbound orbits with effective energy smaller than $V_{\rm eff}(r_{\rm crit})$, which approach the centre of attraction and then move away. Like the hyperbolic orbits of the inverse-square field, these orbits will be symmetrical about the point of closest approach. Orbits with greater effective energy will pass inside $r_{\rm crit}$ and in to $r = 0$. What happens here is perhaps ill-defined. The particle might pop out of the other side and depart again. There is an unstable circular orbit at $r_{\rm crit}$. The final type of orbit has effective energy smaller than $V_{\rm eff}(r_{\rm crit})$, and $r < r_{\rm crit}$. These orbits are bound and involve repeated passages through $r = 0$, at which point what happens is perhaps ill-defined.



Rotating frames

T.26 Vertical. (a) The centrifugal force contributes to what we call the vertical. The angle δ between vertical and the radial direction is given by

$$\frac{\sin\delta}{\omega^2 r} \simeq \frac{\sin(90-\theta)}{g} \tag{198}$$

where r is the radial distance to the axis, which is $R_{\rm e}\sin\theta$, and θ is roughly 38 degrees in Cambridge. So

$$\delta \simeq \frac{\sin\theta\cos\theta\omega^2 R_{\rm e}}{g} \simeq 3\times10^{-3}\,\text{degrees} \quad \text{or } 0.2 \text{ degrees.} \tag{199}$$

($R_{\rm e} \simeq 6\times10^6\,$m; $\omega \simeq 2\pi 10^{-5}\,$s^{-1})

(b) Crude estimate by Coriolis force: the typical velocity is of order $v = \sqrt{gh}$; the fall time is of order $\sqrt{h/g}$; and the typical Coriolis acceleration is of order ωv. When we accelerate at ωv for a time $t = \sqrt{h/g}$, the displacement is of order $(\omega v)t^2 = \omega\sqrt{gh}h/g = \omega h^{3/2}/g^{1/2}$.

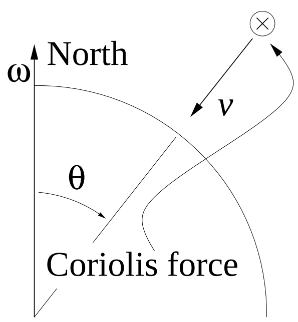
Based on extensive experience of falling out of helicopters, I'd say the effect is not obvious to the eye, so we expect an answer smaller than one metre.

(i) Angular momentum: let R be earth radius, and h the height of helicopter above earth. The initial angular momentum is $J = m(R+h_0)^2\omega$, and the true Eastward velocity (relative to the inertial frame) is $v_{\rm True} = J/[m(R+h_0)]$. As h varies, J is conserved, so we can deduce the true velocity using $v_{\rm True} = J/[m(R+h(t))] = (R+h_0)^2\omega/(R+h(t))$. This can be contrasted with the Eastward velocity of the grid of the rotating frame, which is $v_{\rm Grid} = \omega(R+h)$. When the mass falls, these two velocities, initially equal, become unequal, and there is a slippage between them:

$$\begin{aligned}
\delta v(t) &= v_{\rm True} - v_{\rm Grid} = (R+h_0)^2\omega/(R+h(t)) - \omega(R+h(t)) && (200)\\
&\simeq \omega R\left[1 - \left(\frac{(R+h(t)}{R+h_0}\right)^2\right] && (201)\\
&\simeq \omega R\left[2(h_0-h(t))/R\right]. && (202)\\
&\simeq 2\omega\left[h_0 - h(t)\right]. && (203)
\end{aligned}$$

The total slippage is given by the integral of this velocity,

$$\begin{aligned}
\delta x &= \int dt\, 2\omega\left[h_0 - h(t)\right] && (204)\\
&= \int dt\, 2\omega\frac{1}{2}gt^2 = \frac{1}{3}\omega g t_{\rm max}^3 && (205)
\end{aligned}$$



Now $t_{\rm max} = \sqrt{2h/g}$ so

$$\delta x = \frac{1}{3}\omega g^{-1/2}2^{3/2}h^{3/2} \tag{206}$$

(ii) Solution by Coriolis force. We assume that the Coriolis effect is relatively small so that the motion is very close to radial plunging. The velocity relative to the rotating frame is radial, and the direction of the Coriolis acceleration, $-2\underline{\omega}\times\mathbf{v}$, is thus due East. The vertical velocity, as a function of time, is $v_z = gt$. The magnitude of the Eastward Coriolis acceleration is $2\omega g t$. The Eastward velocity is $v_x = \int dt\, \omega g t = \omega g t^2$. The Eastward displacement is

$$x = \int dt\, v_x = \omega g t^3/3.$$

The fall time t is related to the height fallen h by $h = \frac{1}{2}gt^2$, so $t = (2h/g)^{1/2}$, and

$$x = \omega g (2h/g)^{3/2}/3 = \omega g^{-1/2} h^{3/2} 2^{3/2}/3.$$

Sanity check the inverse dependence on g: if g gets bigger, the stone zips down so fast that rotation of the frame is less noticeable; bigger height gives bigger displacement.

Numerical answer: $\omega \simeq 7 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$, $x = 0.24\,\mathrm{m}$.

T.27 Missile. The velocity is near horizontal and due East. The Coriolis force is directed away from the earth's axis. Coriolis force $= 2m\omega v$ to the right (which means right-and-up from the point of view of the missile, but $\cos\theta \simeq 1$). So the Coriolis acceleration is $a = \omega v \cos\theta$ The flight lasts a duration $t = \frac{D}{v}$, the distance displaced to the South is $x = \frac{1}{2}at^2 = \omega \frac{D^2}{v}\cos\theta \approx 150\,\mathrm{m}$ south for $v = 1000$ m s^{-1} .

Note that the slower the missile, the larger the deflection.

T.28 Circular coordinates. In ordinary inertial coordinates (r, θ),

$$\begin{aligned} L &= T - V && (207)\\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r,\theta) && (208)\end{aligned}$$

The conjugate momenta are

$$p_r = m\dot{r}, \quad p_\theta = mr^2\dot{\theta}. \tag{209}$$

The equations of motion are

$$\frac{d}{dt}m\dot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}, \quad \frac{d}{dt}\left[mr^2\dot{\theta}\right] = -\frac{\partial V}{\partial \theta}. \tag{210}$$

If we use non-inertial coordinates (r, θ) relative to a frame rotating at ω, the kinetic energy becomes

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\dot{\theta} + \omega)^2 \tag{211}$$

The Lagrangian becomes

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\dot{\theta} + \omega)^2 - V(r,\theta) \tag{212}$$

The conjugate momenta are

$$p_r = m\dot{r}, \quad p_\theta = mr^2(\dot{\theta} + \omega). \tag{213}$$

The Euler–Lagrange equation for r is

$$\frac{d}{dt} m\dot{r} = mr(\dot{\theta}+\omega)^2 - \frac{\partial V}{\partial r}, \tag{214}$$

or, in a form convenient for comparison with the ordinary equation of motion (210),

$$\tfrac{d}{dt} m\dot{r} \;=\; mr\dot{\theta}^2 - \tfrac{\partial V}{\partial r} \quad + \quad \underbrace{2m\omega r\dot{\theta}}_{\text{Coriolis}} \quad + \quad \underbrace{mr\omega^2}_{\text{centrifugal}}. \tag{215}$$

There are two extra terms compared with (210). The term labelled Coriolis is the radial component of the Coriolis force; the other term is the centrifugal force.

T.29 Disc.

Moment of inertia of disc about $\perp$ axis:

$$I_z = m\int_0^R \frac{2\pi r dr}{\pi R^2} r^2 = \frac{1}{2} m R^2$$

By $\perp$ axis rule, $I_x = I_y = \frac{1}{4} m R^2$. So the moment of inertia tensor is

$$\mathbf{I} = \begin{pmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{pmatrix} \tag{216}$$

$$\underline{\omega} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \tag{217}$$

$$\mathbf{J} = \mathbf{I}\,\underline{\omega} = \frac{mR^2}{4\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0 \\ 1.4 \end{pmatrix} \times 10^{-3}\ \text{kg m}^2\ \text{s}^{-1} \tag{218}$$

The kinetic energy is

$$T = \frac{1}{2}\underline{\omega}^{\mathrm{T}}\mathbf{I}\underline{\omega} = \frac{1}{2}\underline{\omega}\cdot\mathbf{J} = \frac{3}{16} m R^2 = \frac{3}{4}\ \text{mJ}. \tag{219}$$

T.30 Tile.

The vertical impulse P from the stick changes the linear motion, and sets the tile rotating.

Immediately after collision, the centre of mass has linear velocity $(0, 0, -v)$ given by:

$$P = mu - mv \tag{220}$$

The impulsive couple is related to the angular velocity immediately after collision, $\underline{\omega}$, by

$$\begin{aligned} \begin{pmatrix} -bP \\ aP \\ 0 \end{pmatrix} &= \mathbf{I}\underline{\omega} = \frac{m}{3} \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\ &= \frac{m}{3} \begin{pmatrix} b^2\omega_x \\ a^2\omega_y \\ (a^2+b^2)\omega_z \end{pmatrix} \end{aligned}$$

This gives: $\omega_x = -\frac{3P}{mb}$, $\omega_y = -\frac{3P}{ma}$, $\omega_z = 0$

We can then use energy convervation, since the collision is elastic:

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 \tag{221}$$

Putting everything in terms of P, we get

$$P = \frac{2u}{7m}, \tag{222}$$

and the rest of the result follows. The velocity at the corner is $\mathbf{v} + \underline{\omega} \times \mathbf{r}$, where $\underline{\mathbf{r}} = (-a, -b, 0)$ is the vector from the centre to the corner.

(Interestingly, the point of impact has its velocity reversed. Is this true in general in elastic collisions?)

T.31 Precession of earth.

The bulge of the earth is produced by the centrifugal force making rock at the equator a tiny bit lighter than rock at the poles. We can model the earth as a spinning ball of jelly and assume that it has settled down *so that the surface of the earth is an equipotential of the total potential.* The centrifugal force can be described using a centrifugal potential (per unit mass)

$$V_{\mathrm{c}} = -\frac{1}{2}\omega^2 r^2, \tag{223}$$

where r is the radial disance from the axis. [This should not be confused with the effective potential encountered in the energy method, which included an angular-momentum-related term decreasing as $J^2/(2mr^2)$.] The derivative of this fictitious potential, $-dV_{\mathrm{c}}/dr$, is the centrifugal force, $\omega^2 r$.

We assume the total potential is the centrifugal potential plus a gravitational potential due to the mass of the earth, pretending this mass is concentrated at the centre of the earth. [This pretence will introduce some inaccuracy since the bulge itself has mass, so the gravitational potential will not have spherical symmetry.]

For locations in the plane of the equator,

$$V_{\mathrm{equator}}(r) = -\frac{1}{2}\omega^2 r^2 - GM/r, \tag{224}$$

and for locations along the earth's axis,

$$V_{\mathrm{axis}}(r) = -GM/r. \tag{225}$$

We could also write down a vector expression for the potential that is valid everywhere, but the above scalar functions will be sufficient; the general expression is:

$$V(\mathbf{r}) = -\frac{GM}{|\mathbf{r}|} - \frac{1}{2}(\underline{\omega} \times \mathbf{r})^2 \tag{226}$$

The equipotentials of $-\frac{GM}{|\mathbf{r}|}$ are spheres; the equipotentials of $V(\mathbf{r})$ are slightly flattened spheres. The polar radius R_{p} that has the same potential as a given equatorial radius R_{e} is given by:

$$V_{\mathrm{axis}}(R_{\mathrm{p}}) = V_{\mathrm{equator}}(R_{\mathrm{e}}) \tag{227}$$

that is, letting $R_\mathrm{e} = R_\mathrm{p} + b$, where b is the relatively small bulge thickness,

$$-\frac{GM}{R_\mathrm{p}} = -\frac{GM}{R_\mathrm{p}+b} - \frac{1}{2}\omega^2 R_\mathrm{e}^2 \tag{228}$$

so, Taylor expanding GM/r,

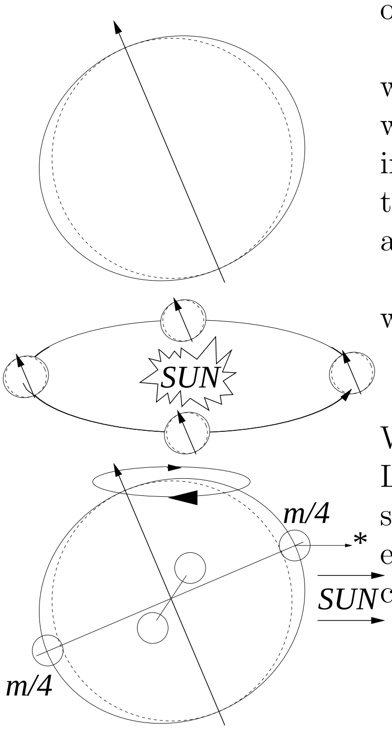
$$bg = \frac{1}{2}\omega^2 R_\mathrm{e}^2 \tag{229}$$

(that is, the potential energy corresponding to the bulge height, bg, is equal to the total centrifugal potential energy for mass going from the north pole to the equator).

$$b = \frac{1}{2} R_\mathrm{e} \frac{\omega^2 R_\mathrm{e}}{g} \tag{230}$$

The ratio $\frac{\omega^2 R_\mathrm{e}}{g}$ s the ratio of the centrifugal force to gravity that we already computed in the vertical question, about 0.002. So we predict the bulge to be 1/200 times R_e.

[In fact, it's a little smaller, because of corrections from the self-gravity of the bulge.]



Here we approximate $\cos 23^o \simeq 1$.

Now, the bulge, which is a tubular hoop of thickness b, radius R_e, and width roughly R_e, is tugged by the gravity of the sun (and the moon, which we'll ignore for starters). The main effect of that tug is to keep the bulge in orbit around the sun, just as the tug on the earth keeps the earth in the same orbit. But a tiny side-effect of the tug on the bulge *may* be, at appropriate seasons of the year, a *torque*.

Now, we can roughly emulate the bulge by four masses of size $m/4$, where

$$m \simeq 2\pi R_\mathrm{e}^2 b \frac{M_\mathrm{e}}{\frac{4}{3}\pi R_\mathrm{e}^3} = \frac{3}{2}\frac{b}{R_\mathrm{e}} M_\mathrm{e} \tag{231}$$

With just four masses, it's easier to compute torques produced by the sun. Let's put the sun off to the right, for starters. The gravitational field strength at distance R from the sun is is $-GM_\mathrm{s}/R^2$, so, differentiating, the excess field strength at the front face of the earth (marked * in the figure), compared with the centre of the earth, is

$$2\frac{GM_\mathrm{s}}{R_\mathrm{s}^3} R_\mathrm{e}, \tag{232}$$

so there is a torque from the front and back masses equal to

$$\frac{dJ}{dt} \simeq 2\frac{GM_\mathrm{s}}{R_\mathrm{s}^3} R_\mathrm{e} \frac{m}{2} \times R_\mathrm{e} \sin 23^o \simeq \frac{GM_\mathrm{s}}{R_\mathrm{s}^3} \frac{3}{2} b M_\mathrm{e} R_\mathrm{e} \sin 23^o. \tag{233}$$

Now, flip the sun to the left hand side (*i.e.*, advance 6 months) – what is the handedness of the torque? It is still the same way, clockwise into the page. If we advance 3 months so the Sun is sideways on (spring and autumn equinoxes) then there is no net torque and $dJ/dt = 0$. Thus the earth precesses as shown in the figure, clockwise as viewed from the North pole. We halve the value for the torque dJ/dt found above to take account of the oscillation of the torque between zero (spring, autumn) and the above value (winter, summer).

The sin 23's cancel.

The precession rate is given by

$$\Omega = \frac{d\bar{J}/dt}{J\sin 23^o} \simeq \frac{GM_\text{s}}{R_\text{s}^3}\frac{3}{4}bM_\text{e}R_\text{e}\frac{1}{I_\text{earth}\omega_\text{earth}} \tag{234}$$

$$\simeq \frac{GM_\text{s}}{R_\text{s}^3}\frac{3}{4}bM_\text{e}R_\text{e}\frac{1}{\frac{2}{5}M_\text{e}R_\text{e}^2\omega_\text{earth}} \tag{235}$$

Substituting for $b = \frac{1}{2}R_\text{e}\omega_\text{earth}^2 R_\text{e}/g$,

$$\Omega \simeq \frac{GM_\text{s}}{R_\text{s}^3}\frac{15}{16}R_\text{e}\frac{\omega^2 R_\text{e}}{g}R_\text{e}\frac{1}{R_\text{e}^2\omega} \tag{236}$$

$$\simeq \frac{GM_\text{s}}{R_\text{s}^3}\frac{15}{16}\frac{\omega R_\text{e}}{g} \tag{237}$$

$$\simeq \frac{15}{16}\omega\frac{\frac{GM_\text{s}}{R_\text{s}^3}}{\frac{GM_\text{e}}{R_\text{e}^3}} \tag{238}$$

$$\simeq \frac{15}{16}\omega\frac{\frac{M_\text{s}}{R_\text{s}^3}}{\frac{M_\text{e}}{R_\text{e}^3}} \tag{239}$$

What an intriguing final answer: the precession rate is the rotation rate of the earth, ω, times the ratio of the density of the solar system, out to radius R_s, to the density of the earth. In terms of more familiar quantities, recall that $GM_\text{s}/R_\text{s}^3 = (2\pi/1\text{ year})^2$, so

$$\Omega \simeq \frac{15}{16}\omega\frac{\frac{GM_\text{s}}{R_\text{s}^3}}{\frac{GM_\text{e}}{R_\text{e}^3}} \simeq \frac{15}{16}\omega\frac{(2\pi/1\text{ year})^2}{\frac{g}{R_\text{e}}} \tag{240}$$

$$\simeq \frac{15}{16}\frac{2\pi}{1\text{ day}}\frac{(2\pi/1\text{ year})^2}{\frac{g}{R_\text{e}}} \simeq \frac{2\pi}{123{,}000\text{ years}} \tag{241}$$

This predicted period of 123,000 years is off by a factor of 5 or 6. The true period is more like 21,000 years; one sign per 2,000 years.

Could we be wrong because we omitted the effect of the moon? The moon and sun make equal almost size tides, and the moon is closer, so the derivative of the gravitational field, which gives the torque, might well be bigger than that for the sun! The moon contributes to (239) an additional term

$$\frac{15}{16}\omega\frac{\frac{M_\text{m}}{R_\text{m}^3}}{\frac{M_\text{e}}{R_\text{e}^3}} \tag{242}$$

where R_m is the distance to the moon, 4×10^8 m. Let's check the contribution of this lunar torque.

$$\frac{15}{16}\omega\frac{\frac{M_\text{m}}{R_\text{m}^3}}{\frac{M_\text{e}}{R_\text{e}^3}} \simeq 4\times 10^{-8}\omega \tag{243}$$

Oops! The sun only gave us $2\times 10^{-8}\omega$, so the lunar effect is twice as big! This means we must boost our precession rate by a factor of nearly 3. We find

$$\text{Corrected period of precession} = 43{,}000\,\text{years}. \tag{244}$$

That is much better! Less than a factor of two wrong.

Why are we still wrong (precessing too slowly)? The easiest way out may be the integral over the bulge, which we treated using four masses. It's possible a factor of 2 could emerge from that.

[Further complications that could be included: the moon's orbit is at 5 degrees to that of the earth round the sun, so the two sin 23's that cancelled above should be replaced by a sin 23 and an oscillating sin 23 ± 5.]

[Could Jupiter also contribute significantly? No, its mass is $M_{\rm s}/1000$, and its closest distance from the earth is bigger than $R_{\rm s}$.]

[Maybe the *tides* also contribute significantly to precession? The height of the midocean tide due to the moon can be estimated to be about 1 foot. That's a lot smaller than the equatorial bulge, which is 20 kilometres.]

[The density of the earth is non-uniform; the light scum continents float on top.]

[For further work on this problem by others, see the website.]

Any corrections to this question sheet, or queries? Please use the automated FAQ system on the website,

`http://www.inference.phy.cam.ac.uk/teaching/dynamics/`,

or send me email.

DJCM. January 23, 2002