NATURAL SCIENCES TRIPOS   Part III

Tuesday 25 April 2006   9.00 to 12.00

EXPERIMENTAL AND THEORETICAL PHYSICS
Minor Topics

*This paper contains sections corresponding to each of the minor topics papers. Candidates who have opted to replace **one** section by either the Entrepreneurship option **or** a project undertaken during the previous Long Vacation should attempt **two** sections and will leave after **two** hours. Candidates who have opted to replace **two** sections by either the Advanced Quantum Field Theory paper, or both a Long Vacation Project **and** the Entrepreneurship option should attempt **one** section and will leave after **one** hour. The remaining candidates should attempt **three** sections.*

*Each section contains three questions, of which you are required to answer **two**. The approximate number of marks allotted to each part of a question is indicated in the right margin where appropriate. The paper contains **28** sides including this one and is accompanied by a booklet giving values of constants and containing mathematical formulae which you may quote without proof.*

You should use a separate booklet for each section.

STATIONERY REQUIREMENTS
3 × 20 Page Answer Book
Rough Work Pad
Metric Graph Paper

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

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SECTION A   GAUGE FIELD THEORY

1   Write brief accounts of **two** of the following:

(a) the Klein paradox in relativistic quantum mechanics;   [10]

(b) the parameters of the Standard Model of electroweak interactions;   [10]

(c) the fermion mass problem;   [10]

(d) the dimensions of quantum fields and their couplings.   [10]

2   Explain what is meant by *second quantization.*   [2]
The Fourier representation of a Klein-Gordon field has the form

$$\hat{\phi}(\boldsymbol{r}, t) = \int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3 2\omega} \left[\hat{a}(\boldsymbol{k}) \, \mathrm{e}^{-ik\cdot x} + \hat{b}^\dagger(\boldsymbol{k}) \, \mathrm{e}^{+ik\cdot x} \right] ,$$

where $\omega = \sqrt{|\boldsymbol{k}|^2 + m^2}$, while that for a Dirac field is

$$\hat{\psi}(\boldsymbol{r}, t) = \int \frac{\mathrm{d}^3 \boldsymbol{k}}{(2\pi)^3 2\omega} \sum_s \left[\hat{c}_s(\boldsymbol{k}) u_s(\boldsymbol{k}) \, \mathrm{e}^{-ik\cdot x} + \hat{d}_s^\dagger(\boldsymbol{k}) v_s(\boldsymbol{k}) \, \mathrm{e}^{+ik\cdot x} \right] .$$

Explain the meaning of the operators $\hat{a}(\boldsymbol{k})$, $\hat{b}^\dagger(\boldsymbol{k})$, $\hat{c}_s(\boldsymbol{k})$ and $\hat{d}_s^\dagger(\boldsymbol{k})$.   [4]

Write down expressions for conserved charge operators for the Klein-Gordon and Dirac fields and explain how they depend on the algebraic properties of the relevant operators.   [8]

Show how to construct two-particle states in each case and how their properties under exchange of the particles follow from those of the relevant operators.   [4]

Demonstrate the effect of the Klein-Gordon charge operator on the corresponding two-particle state.   [2]

3   A complex scalar field ϕ has the Lagrangian density

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) + \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 .$$

Discuss why additional higher-order terms are omitted from the right-hand side.   [4]

Give an account of the quantum excitations of the field and their properties.   [6]

Explain how these properties are modified if the scalar field couples to an Abelian gauge field, B^μ, with coupling strength g.   [4]

Show that, under these circumstances, the quanta of the gauge field acquire a mass, and find its value.   [2]

Show that there is a coupling between a quantum of the scalar field and two gauge field quanta, and that the corresponding decay process is kinematically possible if $\lambda > 2g^2$.   [4]

SECTION B   INFORMATION THEORY, PATTERN
RECOGNITION AND NEURAL NETWORKS

1   A channel has a 3-bit input,

$$x \in \{000, 001, 010, 011, 100, 101, 110, 111\},$$

and a 2-bit output $y \in \{00, 01, 10, 11\}$. Given an input x, the output y is generated by *deleting* exactly one of the three input bits, selected at random. For example, if the input is $x = 010$ then $P(y \mid x)$ is 1/3 for each of the outputs 00, 10, and 01; if the input is $x = 001$ then $P(y=01 \mid x) = 2/3$ and $P(y=00 \mid x) = 1/3$.

Write down the conditional entropies $H(Y \mid x{=}000)$, $H(Y \mid x{=}010)$, and $H(Y \mid x{=}001)$.   [3]

Assuming an input distribution of the form

x	000	001	010	011	100	101	110	111
$P(x)$	$\dfrac{1-p}{2}$	$\dfrac{p}{4}$	0	$\dfrac{p}{4}$	$\dfrac{p}{4}$	0	$\dfrac{p}{4}$	$\dfrac{1-p}{2}$,

where $p \in (0, 1)$, work out the conditional entropy $H(Y \mid X)$ and show that

$$H(Y) = 1 + H_2 \left(\frac{2}{3}p \right),$$

where $H_2(x) = x \log_2(1/x) + (1 - x) \log_2(1/(1 - x))$.   [3]

Sketch $H(Y)$ and $H(Y \mid X)$ as a function of $p \in (0, 1)$ on a single diagram.   [5]

Sketch the mutual information $I(X; Y)$ as a function of p.   [2]

$\left[\, H_2(1/3) \simeq 0.92. \, \right]$

Another channel with a 3-bit input

$$x \in \{000, 001, 010, 011, 100, 101, 110, 111\},$$

erases exactly one of its three input bits, marking the erased symbol by a ?. For example, if the input is $x = 010$ then $P(y \mid x)$ is 1/3 for each of the outputs ?10, 01?, and 0?0.

What is the capacity of this channel? Describe a method for reliable communication over it.   [7]

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2   Three six-sided dice have faces labelled with symbols as follows:

Name of die	Number of faces having symbol					
	r	s	t	u	v	w
A	1	1	1	1	1	1
B	0	3	3	0	0	0
C	2	1	2	1	0	0

For example, die B has 3 faces labelled s and 3 faces labelled t.

One of the dice is selected at random and is rolled N times, creating a sequence of outcomes $x_1, x_2, \ldots x_N$. The identity of the selected die, d, is not explicitly revealed.

(a) What is the probability distribution of the first outcome, x_1? Describe an optimal binary symbol code for encoding the first outcome. [6]

(b) Assume that the first outcome is $x_1 = $ s. Given this information, how probable are the alternative theories about which die was chosen, $d = $ A, $d = $ B, $d = $ C? [3]

(c) Given that $x_1 = $ s, what is the probability distribution of the second outcome x_2? [4]

(d) Assume the the entire sequence of N outcomes $\boldsymbol{x} = x_1, x_2, \ldots x_N$ is compressed by arithmetic coding using the appropriate predictive distribution $P(x_n \mid x_1, \ldots, x_{n-1})$. Sketch the probability distribution of the compressed file's length $l(\boldsymbol{x})$. You may assume that $N \gg 100$. [*You may find the following facts useful:* $H_2(1/3) \simeq 0.92$; $\log_2 6 \simeq 2.6$.] [7]

3   Decay events occur at distances $\{x_n\}$ from a source. Each distance x_n has an exponential distribution with characteristic length λ. If λ were known, the probability distribution of x would be

$$P(x \mid \lambda) = \frac{1}{\lambda} e^{-x/\lambda}.$$

The locations of decay events that occur in a window from $x = 0$ to $x = b$ are measured accurately. Decay events at locations $x_n > b$ *are also detected*, but the actual *value* of x_n is not obtained in those cases. The probability of such an overflow event is

$$P(x > b \mid \lambda) = \int_b^\infty \frac{1}{\lambda} e^{-x/\lambda} \, \mathrm{d}x = e^{-b/\lambda}.$$

In an experiment where the right hand side of the window is at $b = 10$ units, a data set of $N = 50$ events is obtained. Of these, $N_< = 9$ events occur in the window $0 \le x \le b$, and $N_> = 41$ events have $x_n > b$. The 9 events in the window were at locations

$$\{0.1,\ 1.2,\ 2.0,\ 3.9,\ 4.3,\ 5.7,\ 6.6,\ 7.4,\ 8.8\},$$

as illustrated in the figure below.
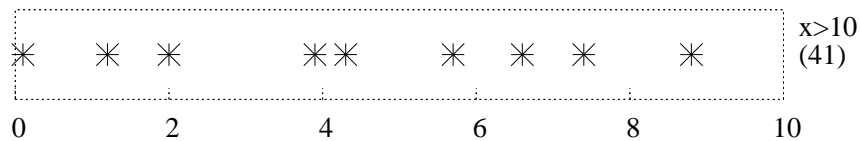
(The sum of these numbers is 40.0.)

(a) Write down the likelihood function and find the maximum-likelihood setting of the parameter λ.   [7]

(b) Sketch the logarithm of the likelihood function, as a function of $\ln \lambda$.   [3]

(c) Find error bars on $\ln \lambda$.   [5]

(d) Imagine that we must choose a follow-up experiment to measure λ more accurately. There are two choices, with identical cost: either (A) the window size can be increased from $b = 10$ to $b = 200$ units, and $N' = 250$ new events can be observed; or (B) the window size can be left where it is, at $b = 10$ units, and a greater number, $N' = 5000$, of events can be observed.

Discuss which of these would be the more informative experiment.   [5]

SECTION C   GENERAL RELATIVITY

1   Write brief notes on **two** of the following:

(a) the equivalence principle and local inertial coordinates;   [10]

(b) the formation of a black hole as perceived by a distant observer;   [10]

(c) the structure of a Kerr black hole;   [10]

(d) energy-momentum tensors.   [10]

2   For a weak gravitational field, there exist coordinate systems x^μ in which the spacetime metric is static and takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $|h_{\mu\nu}| \ll 1$. In such a coordinate system, show that, to first order in $h_{\mu\nu}$, the connection coefficients $\Gamma^\mu{}_{00}$ are given by

$$\Gamma^0{}_{00} = 0 \qquad \text{and} \qquad \Gamma^i{}_{00} = \tfrac{1}{2}\delta^{ij}\partial_j h_{00},$$

where $i, j = 1, 2, 3$.   [3]

In a weak gravitational field, consider a massive particle moving with coordinate velocity $u^i = \mathrm{d}x^i/\mathrm{d}t$, where $ct = x^0$. If the speed u of the particle is much less than c, show that to zeroth order in u/c the equation of motion of the particle is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\tfrac{1}{2}c^2\delta^{ij}\partial_j h_{00},$$

and thus obtain a relation between h_{00} and the Newtonian gravitational potential Φ in this limit.   [6]

Now working to first order in u/c, show the equation of motion of the particle becomes

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\tfrac{1}{2}c^2\delta^{ij}\partial_j h_{00} - c\delta^{ik}(\partial_k h_{0j} - \partial_j h_{0k})u^j.$$

[7]

Show that this equation can be written in 3-vector form as

$$\frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}t^2} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B},$$

where $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi$, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and the ith component of $\boldsymbol{A}$ is given by ch^{0i}. Briefly compare this result with the Lorentz force law in electrodynamics.   [4]

$\left[\textit{You may assume that } \Gamma^\sigma{}_{\mu\nu} = \tfrac{1}{2}g^{\sigma\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}). \right]$

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3   By considering the effective Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, or otherwise, show that the geodesic equations for a massive particle may be written as

$$\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = \tfrac{1}{2}(\partial_\mu g_{\rho\sigma})u^\rho u^\sigma,$$

where u_μ are the covariant components of its 4-velocity and τ is its proper time.   [3]

The Schwarzschild metric is

$$\mathrm{d}s^2 = c^2\left(1 - \frac{2\mu}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2.$$

Show that for a massive particle in a circular orbit of coordinate radius r in the equatorial plane,

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{\mu c^2}{r^3}\right)^{1/2}.$$

[4]

A particle in an accretion disc around a central mass follows a circular orbit of radius r in the equatorial plane of the Schwarzschild geometry with coordinate angular velocity Ω. The particle emits a photon that is received by an observer who is stationary at infinity in the direction $\phi = 0$ in the equatorial plane. Show that, in general, the ratio of the photon frequency at reception, ν_R, to that at emission, ν_E, is given by

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3\mu}{r}\right)^{1/2}\left[1 + \frac{p_3(E)}{p_0(E)}\Omega\right]^{-1},$$

where $p_0(E)$ and $p_3(E)$ are the zeroth and third covariant components, respectively, of the photon's 4-momentum at emission.   [7]

If the photon is emitted when the particle motion is transverse to the observer, i.e. when $\phi = 0$ or $\phi = \pi$, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3\mu}{r}\right)^{1/2}.$$

[2]

If the photon is emitted when the particle is moving either directly towards or away from the observer, i.e. when $\phi = -\pi/2$ or $\phi = \pi/2$ respectively, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3\mu}{r}\right)^{1/2}\left[1 \pm \left(\frac{r}{\mu} - 2\right)^{-1/2}\right]^{-1},$$

identifying clearly to which case the plus and minus signs correspond.   [4]

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SECTION D   THE FRONTIERS OF PARTICLE PHYSICS

1   A possible decay chain for a supersymmetric quark is

$$\tilde{u}_L \to q\tilde{\chi}_2^0 \to qe^{\pm}\tilde{e}_R^{\mp} \to qe^{\pm}e^{\mp}\tilde{\chi}_1^0 .$$

Identify the supersymmetric particles in this decay chain, giving their Standard Model partners, electric charges, and spins. In the case of mixed states, specify at least one component. For each scalar particle, also specify the weak isospin.   [5]

In a two-body decay, the momentum of the outgoing particles in the rest frame of the parent is given by

$$p = \frac{m_1^2 - m_2^2}{2m_1} ,$$

where m_1 is the mass of the parent, m_2 is the mass of one of the outgoing particles, and the mass of the other outgoing particle is negligible. Using this result, compute the momenta of the electrons in the above decay chain, in the rest frame of the $\tilde{e}_R$.   [4]

Hence show that the invariant mass of the e^+e^- pair cannot exceed   [5]

$$m_{max}^{ee} = \frac{1}{m_{\tilde{e}_R^{\mp}}} \sqrt{(m_{\tilde{\chi}_2^0}^2 - m_{\tilde{e}_R^{\mp}}^2)(m_{\tilde{e}_R^{\mp}}^2 - m_{\tilde{\chi}_1^0}^2)} .$$

Explain the importance of R-parity conservation in experimental searches for supersymmetry. List the key experimental signatures expected in events containing supersymmetric particles at the LHC, and briefly describe the experimental apparatus used to detect each signature.   [6]

2   Write an essay on *theories with extra space dimensions*. Your answer should include an account of the motivations for the consideration of such theories, potential signatures, and possible methods of experimental detection of extra dimensions.   [20]

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3   The Higgs Lagrangian has the form

$$L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V \,,$$

where $V = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda^2\phi^4$, and ϕ is the Higgs field strength and μ and λ are constants. Show that the field has a vacuum expectation value of $\pm\mu/\lambda$.   [2]

     By expanding the field about the minimum, show that the Higgs mass is predicted to be $\sqrt{2}\mu$. Identify the other terms in the Lagrangian.   [4]

$$\left[\begin{array}{l} \textit{The Klein-Gordon Lagrangian for a scalar field } \phi \textit{ is given by} \\ \quad L = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2. \end{array}\right]$$
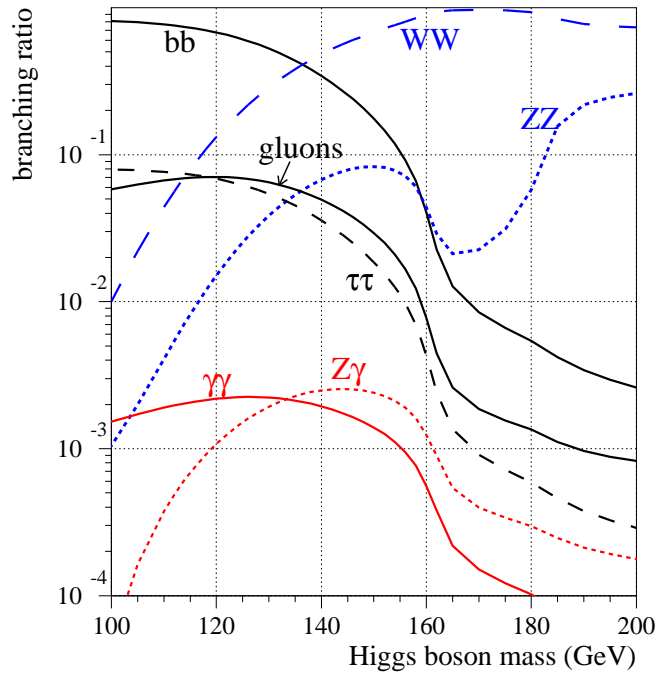
     The figure shows the principal branching ratios of the Higgs boson as a function of its mass. Explain the behaviour of the branching ratios into b-quarks, W and Z bosons.   [5]

     Consider the possible decay modes of a Higgs boson with a mass of $200\,\text{GeV}$, and explain briefly which mode would give the best chance of discovery. Describe the apparatus needed to detect the *final-state* particles for this mode.   [6]

     If no candidate Higgs events are observed after the experiment has recorded an integrated luminosity of $1\ \text{fb}^{-1}$, what is the 95% confidence limit on the product of the production cross-section and the decay branching ratio?   [3]

$$\left[\begin{array}{l} \textit{The probability of observing } n \textit{ events when sampling from a distribution with} \\ \textit{mean } \mu \textit{ is given by } P(n) = \frac{\mu^n}{n!}e^{-\mu}. \end{array}\right]$$

SECTION E   SUPERCONDUCTIVITY AND QUANTUM COHERENCE

1   The mean-field Hamiltonian for wavefunctions close to the BCS ground state is given by

$$\hat{H} = \sum_{\boldsymbol{k}\sigma} \epsilon_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \sum_{\boldsymbol{k}} (\Delta^{\star}_{\boldsymbol{k}} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} + \Delta_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\uparrow} c^{\dagger}_{-\boldsymbol{k}\downarrow}) \,,$$

where $\epsilon_{\boldsymbol{k}}$ is the single-electron energy with respect to the Fermi energy and $\Delta_{\boldsymbol{k}}$ is the gap parameter. For a d-wave high-$T_{\rm c}$ superconductor, $\Delta_{\boldsymbol{k}} \simeq \Delta_0 \cos 2\phi$, where ϕ is the azimuthal angle in the (k_x, k_y)-plane and, for simplicity, we take Δ_0 to be a real constant.

(a) Describe, in words, the origin of this Hamiltonian and give a pictorial description of the terms in $\hat{H}$.   [4]

(b) The Bogoliubov transformation introduces the fermionic operators $b^{\dagger}_{\boldsymbol{k}\uparrow} = u_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}} c_{-\boldsymbol{k}\downarrow}$ and $b^{\dagger}_{\boldsymbol{k}\downarrow} = u_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}} c_{-\boldsymbol{k}\uparrow}$, where $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ are real and positive with $2u_{\boldsymbol{k}}^2 = 1 + \dfrac{\epsilon_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}$, $2v_{\boldsymbol{k}}^2 = 1 - \dfrac{\epsilon_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}$, and $E_{\boldsymbol{k}}^2 = \epsilon_{\boldsymbol{k}}^2 + \Delta_{\boldsymbol{k}}^2$. Interpret the physical nature of the fermions $b_{\boldsymbol{k}\sigma}$ and $b^{\dagger}_{\boldsymbol{k}'\sigma'}$, and show explicitly that the anticommutator between $b_{\boldsymbol{k}\sigma}$ and $b^{\dagger}_{\boldsymbol{k}'\sigma'}$ gives the result expected for fermions.   [5]

(c) In terms of these new operators, $\hat{H}$ can be rewritten as

$$\hat{H} = \sum_{\boldsymbol{k}\sigma} E_{\boldsymbol{k}} b^{\dagger}_{\boldsymbol{k}\sigma} b_{\boldsymbol{k}\sigma} + \text{const.}$$

Without deriving this result, comment on its significance in the context of the BCS theory of superconductivity.   [3]

(d) The Fermi surface of the high-$T_{\rm c}$ cuprates may be approximated by a cylinder of radius $k_{\rm F}$ and height $h_{\rm BZ}$, with a uniform Fermi velocity $v_{\rm F}$. Show that the dispersion relation of $E_{\rm k}$ vs. k_x and k_y has equal energy contours which, for small energies E, form ellipses around the gap nodes with semi-major and semi-minor axes $\dfrac{E k_{\rm F}}{2\Delta_0}$ and $\dfrac{E}{\hbar v_{\rm F}}$, respectively.   [4]

(e) Use the previous result to show that the density of states in the high-$T_{\rm c}$ superconductors is linear in E for small E, and determine the proportionality factor. Give an example of how one can experimentally verify this linear relationship.   [4]

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2   (a) The free energy of a superconducting weak link is given by $F = -F_0 \cos \phi$, where ϕ is the phase difference across the link. Justify this expression, and state how and why the definition of ϕ needs to be amended when the link is placed in a magnetic field. [3]

(b) From this expression for the free energy, deduce the governing equation of the weak link that relates the current I through the link to the voltage, V, and phase difference, ϕ, across the link. [3]

(c) Using the washboard analogy, without detailed calculation, describe and sketch the DC I-V characteristic expected for a weak link driven by a pure DC current. [3]

(d) The governing equation also applies to superfluid weak links, I being the volume current in that case. We consider a vessel of superfluid ^{3}He-B as sketched below, involving two weak links that each have Josephson critical current I_J.
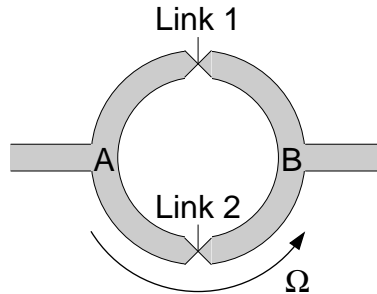

The whole assembly is rotated with respect to the rest frame at angular velocity, Ω, around the normal to the loop. The superflow through the links is relatively small, so the bulk of the superfluid can be taken to rotate at Ω also. By analogy with a SQUID, by considering the circulation integral around a loop, show that the total superflow through the two links is given by

$$I_s = 2I_J \cos \left(\frac{2m_3 S \Omega}{\hbar} \right) \sin \bar{\phi},$$

where S is the area of the loop, m_3 is the mass of one ^{3}He atom, and $\bar{\phi}$ is the phase difference between the points A and B. [6]

(e) Experimentally, this assembly was first realised by a research group at Berkeley. The loop had an area $S = 6 \, \text{cm}^2$. The normal to the loop lay in a horizontal plane at an angle α relative to the East direction. The graph overleaf shows measurements of the critical current through the two links as a function of α. Use the data to determine the geographical latitude of Berkeley. [5]

Area normal
east/west

Area normal
north/south
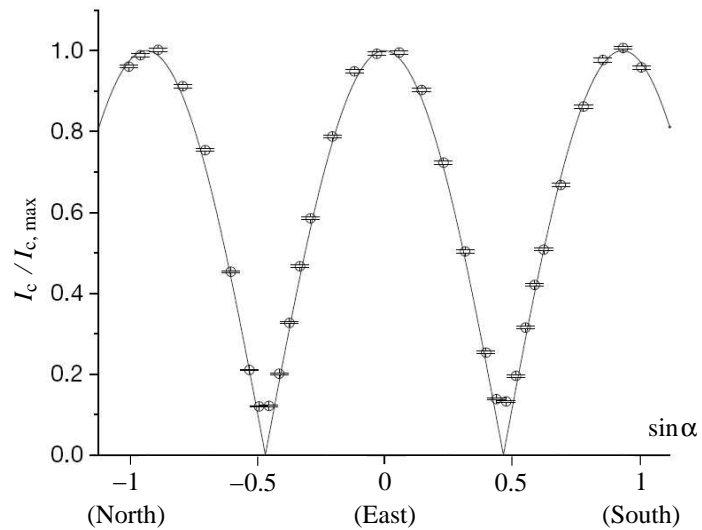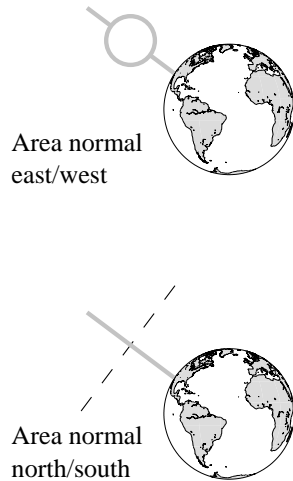
3   Write brief notes on **two** of the following:

(a) Most superconducting elements are found to be type-I superconductors, while most superconducting compounds with elevated values of T_c are type-II.   [10]

(b) Superfluid ^{4}He is an extremely good thermal conductor, while the thermal conductivity of superconductors is strongly suppressed far below T_c.   [10]

(c) Structural defects are intentionally introduced into superconducting wires in electromagnets, while one usually tries to remove defects in normal metallic conductors.   [10]

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SECTION F   QUANTUM ELECTRONICS IN
        SEMICONDUCTORS

1   State the assumptions of the Landauer transport formalism.   [4]
        Calculate the conductance of a one-dimensional quantum dot in terms of the
reflection and transmission amplitudes r and t of its two tunnel barriers
(considered identical) and the phase ϕ acquired in passing from one side of the dot
to the other.   [3]
        Give a justification for expecting to see a transmission resonance associated
with each eigenstate of the quantum dot.   [1]
        Sketch and explain the features of the single-particle eigenspectrum of a
two-dimensional parabolic quantum dot.   [3]
        Use the quantum Coulomb blockade model to justify why you would not
expect the peak-position trajectories of a quantum dot simply to map out its
single-particle eigenspectrum as a function of dot gate voltage V_g and magnetic
field, B.   [5]
        Draw the expected V_g $versus$ B peak trajectories and explain the differences
from the dot single-particle eigenspectrum.   [2]
        Explain what pattern you would expect the addition energies of a
two-dimensional parabolic quantum dot to follow as a function of particle number.   [2]

2   Draw and label the conduction-band edge of a GaAs-AlGaAs heterostructure
containing a single two-dimensional electron gas.   [3]
        Explain what is meant by the 'effective potential' in a two-dimensional
electron gas and how it can be modulated using metal surface gates.   [2]
        Explain the general principles behind the detection of ballistic propagation
of electrons.   [4]
        Derive an expression for the classical 'refraction' that occurs when a ballistic
electron passes at an angle between two regions of different carrier density in a
two-dimensional electron gas.   [2]
        Ballistic electrons in a two-dimensional electron gas of carrier density
$n_2 = 1.0 \times 10^{15}$ m^{-2} are incident normally on one surface of a wedge-shaped region
of lower density n_W and internal angle $\theta = 60°$. By calculating the exit angle θ' for
a small number of values of n_W produce a sketch of θ' $versus$ n_W.   [4]
        How could this device be used as a ballistic-electron switch?   [2]
        What would limit the accuracy and speed of operation of such a switch?   [3]

3   Explain what is meant by band bending in a semiconductor electronic device. [3]

     Consider a GaAs-AlGaAs heterostructure consisting of two identical quantum wells separated by an AlGaAs barrier of 125 Å thickness. Each quantum well contains a single two-dimensional electron gas so that the system can be used in tunnelling experiments.

     Sketch the conduction-band edge of this device in just the double-well region for the following cases: (i) front well empty; (ii) front and back well equal in density; and (iii) front well much higher in density than the back. [3]

     Sketch the carrier densities of the front and back wells and their sum as a function of surface-gate voltage and give justifications for these dependences. [3]

     Make sketches of the Fermi-surface spectral function of each of the two two-dimensional electron gases in order to explain how they depend on carrier density and on an external magnetic field $\boldsymbol{B} = (B_x, B_y, 0)$ applied parallel to the surface of the device. [3]

     Describe how magneto-tunnelling spectroscopy can be used to probe the Fermi-surface spectral function of a two-dimensional electron gas. [4]

     Sketch and explain the features of the equilibrium differential tunnelling conductance $G = \mathrm{d}I/\mathrm{d}V$ between two two-dimensional electron gases as a function of surface-gate voltage at a fixed external parallel magnetic field. [3]

     How could magneto-tunnelling measurements be used to check for non-parabolicity in the two-dimensional dispersion relations? [1]

SECTION G   FROM QUANTUM OPTICS TO QUANTUM
            MATTER

1   Explain how correlation measurements can be used to reveal non-classical features of optical fields.   [6]
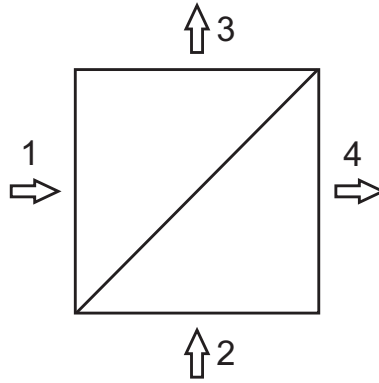
A beamsplitter has two input ports, labelled 1 and 2, and two output ports, labelled 3 and 4, as shown in the figure. The operator relations between the input and output bosonic mode operators are:

$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2 \,,$$
$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2 \,.$$

(a) Show that if the input mode operators are independent, the same is true of the output mode operators.   [3]

(b) Find the output state of the system when input mode 1 contains one photon. If an observation of the output finds a vacuum state in port 3, what is the final state of the output?   [5]

(c) Find the expectation value $\langle \hat{a}_3^\dagger \hat{a}_3 \hat{a}_4^\dagger \hat{a}_4 \rangle$, and explain what this quantity means. How does this quantum picture of this quantity differ from the classical analogue?   [6]

$\left[\textit{Note that energy conservation requires } |R|^2 + |T|^2 = 1 \textit{ and } R^*T + T^*R = 0.\right]$

2   Write brief notes on **two** of the following:

(a) the consequences of the description of a Bose-Einstein condensate by a macroscopic wavefunction;   [10]

(b) the tuning of effective scattering lengths by a Feshbach resonance;   [10]

(c) coherent states for bosons and for paired fermions, and the BEC-BCS crossover.   [10]

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3   In the description of diffraction-limited propagation, an electromagnetic wave of wavelength λ is commonly described by the parameter q, defined such that $\frac{1}{q} = \frac{1}{R} + i\left(\frac{\lambda}{\pi w^2}\right)$, where R is the radius of wavefront curvature and w the width of the beam waist. Wave propagation is then treated by modelling each element of an optical system by an 'ABCD' matrix; the effect on q is then given by:

$$q_{\text{out}} = \frac{Aq_{\text{in}} + B}{Cq_{\text{in}} + D}.$$

(a) Explain why one condition for the formation of a stable laser mode in the cavity is that, after one round trip, the beam should have the same value of q.   [2]

(b) Using the condition on q for mode stability show that, in terms of the ABCD matrix elements,   [2]

$$B\left(\frac{1}{q}\right)^2 + (A - D)\left(\frac{1}{q}\right) - C = 0.$$

(c) By comparison of the real and imaginary parts of the solution for $\frac{1}{q}$, and noting that for such a cavity the ABCD matrix is unimodular (so that $AD - BC = 1$) show that, for the cavity to support a stable mode,   [5]

$$\left|\frac{A + D}{2}\right| \leq 1.$$

(d) For a symmetrical linear cavity consisting of two concave mirrors of radius of curvature R, separated by a distance L, derive the ABCD matrix describing a single round trip in the cavity,   [6]

(e) For what range of mirror separation, L, will this cavity support stable longitudinal modes?   [5]

SECTION H   PHASE TRANSITIONS AND COLLECTIVE PHENOMENA

1   In the restricted solid-on-solid model, the Hamiltonian of a rough surface is specified by

$$H = K \sum_{\langle \boldsymbol{lm} \rangle} |h_{\boldsymbol{l}} - h_{\boldsymbol{m}}|^{\infty},$$

where the discrete coordinates $\boldsymbol{l}$ and $\boldsymbol{m}$ each index the sites of a two-dimensional square lattice, and the height variable $h_{\boldsymbol{l}}$ can take positive and negative integer values. Here we have used the notation $\langle \boldsymbol{lm} \rangle$ to indicate that the sum involves only neighbouring sites of the lattice.

(a) Considering βH, where $\beta = \dfrac{1}{k_{\mathrm{B}} T}$ with T the temperature, show that the height difference between neighbouring sites can only assume values of ± 1 or zero.   [3]

(b) As a consequence, taking the boundary conditions to be periodic, show that the $N \times N$ site Hamiltonian may be recast in terms of the $2 \times N \times N$ variables $n_{\boldsymbol{lm}} = h_{\boldsymbol{l}} - h_{\boldsymbol{m}}$ indexing the bonds between neighbouring sites. Explain why the sum of $n_{\boldsymbol{lm}}$ around each square plaquette (i.e. unit cell boundary) of the lattice is constrained to be zero, i.e. defining $\hat{\boldsymbol{e}}_x = (1, 0)$ and $\hat{\boldsymbol{e}}_y = (0, 1)$, for each lattice site $\boldsymbol{l}$,   [4]

$$n_{\boldsymbol{l}, \boldsymbol{l}+\hat{\boldsymbol{e}}_x} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_x, \boldsymbol{l}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_y, \boldsymbol{l}+\hat{\boldsymbol{e}}_y} + n_{\boldsymbol{l}+\hat{\boldsymbol{e}}_y, \boldsymbol{l}} = 0 \,.$$

(c) Imposing these constraints using the identity $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{\pm in\theta} = \delta_{n0}$ for integer n, show that the partition function can be written as   [6]

$$\mathcal{Z} = \left(\prod_{\boldsymbol{l}} \int_0^{2\pi} \frac{\mathrm{d}\theta_{\boldsymbol{l}}}{2\pi} \right) \exp \left\{ \sum_{\langle \boldsymbol{lm} \rangle} \ln \left[1 + 2\mathrm{e}^{-\beta K} \cos(\theta_{\boldsymbol{l}} - \theta_{\boldsymbol{m}}) \right] \right\} \,.$$

(d) At low temperatures (i.e. $\beta K \gg 1$), show that the system becomes equivalent to that of the classical two-dimensional XY spin model. Without resorting to detailed calculation, discuss the significance of this correspondence for the phase behaviour of the restricted solid-on-solid model?   [7]

2   Write a detailed essay on **one** of the following topics:

(a) the scaling theory of critical phenomena;   [20]

(b) Goldstone modes and the lower critical dimension;   [20]

(c) the upper critical dimension and the Ginzburg criterion.   [20]

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3   Starting with the Ginzburg-Landau Hamiltonian for a d-dimensional system,

$$\beta H = \int \mathrm{d}^d x \left[\frac{t}{2} m^2 + \frac{K}{2} (\boldsymbol{\nabla} m)^2 - hm + \frac{L}{2} (\boldsymbol{\nabla}^2 \phi)^2 + v \boldsymbol{\nabla} m \cdot \boldsymbol{\nabla} \phi \right] ,$$

involving the two one-component fields $m(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$:

(a) recast βH in terms of the Fourier elements $m(\boldsymbol{q}) = \int \mathrm{d}^d x \, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} m(\boldsymbol{x})$ and

$\phi(\boldsymbol{q}) = \int \mathrm{d}^d x \, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} \phi(\boldsymbol{x})$.   [3]

(b) Construct a renormalisation group transformation by rescaling distances such that $\boldsymbol{q}' = b\boldsymbol{q}$, and the fields such that $m'(\boldsymbol{q}') = m(\boldsymbol{q})/z$ and $\phi'(\boldsymbol{q}') = \phi(\boldsymbol{q})/y$.   [8]

(c) At the fixed point $K' = K$ and $L' = L$, obtain the exponents y_t, y_h and y_v.   [3]

(d) The singular part of the free energy has a scaling form $f(t, h, v) = t^{2-\alpha} g(h/t^\Delta, v/t^w)$ for t, h and v close to zero. Find α, Δ and w.   [3]

(e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point and how do they scale?   [3]

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SECTION I   SHOCK WAVES AND EXPLOSIVES

1   A metal is to be characterised using a plate-impact facility. Describe, with the help of appropriate diagrams, the significance of the following parameters and the techniques that can be used to obtain them:

   (a) the Hugoniot;   [6]

   (b) the Isentrope;   [3]

   (c) the Hugoniot Elastic Limit;   [3]

   (d) the Spall Strength.   [6]

   If the material undergoes a shock-induced phase transition, indicate the change that can be seen in stress-gauge traces as a result of this process.   [2]

2   Write notes on **two** of the following phenomena associated with explosives:

   (a) An explosive can burn rapidly (deflagrate) or can react via a shock wave mechanism (detonate). Describe the physical mechanisms involved in the transition from deflagration to detonation in a granular bed of such explosive.   [10]

   (b) Reaction in explosives can be produced by mechanical stimulus. Describe the physical processes by which the energy is concentrated to produce reaction. Include reference to liquid, solid and granular systems.   [10]

   (c) Chapman-Jouget (CJ) and Zeldovich-von Neumann-Doering (ZND) theories are widely used to describe detonation. Outline the similarities and differences between the two models, indicating the significance of the Hugoniots of the explosive and its products.   [10]

3   A ballistic experiment is to be conducted to characterise the interaction between a metal rod normally incident on a transparent ceramic plate.
   Draw a diagram indicating the variety of phenomena that can occur in the penetration process.   [6]
   Briefly describe in each case one technique that could be used to observe:

   (a) the fracture in the ceramic plate;   [3]

   (b) the flexing of the impactor;   [3]

   (c) the in-plane motion (i.e. transverse to the direction of impact) of the rear of the plate.   [3]

   The penetration may be assumed to be hydrodynamic. Using the Bernoulli equation, or otherwise, derive a formula relating the densities of the materials to the depth of material penetrated.   [5]

SECTION J   THE FRONTIERS OF EXPERIMENTAL
        ASTROPHYSICS

1   Explain what is meant by the terms *completeness* and *reliability* of a survey.   [4]
In a given flux-limited survey of a population of sources at some known distance, for example a study of the globular clusters in a nearby elliptical galaxy or the stars in an open cluster, how might the survey be planned so as to optimize both its completeness and reliability.   [4]
Assume that the luminosity function for stars in the local neighbourhood of the Solar System can be written as $\Phi(L)\mathrm{d}L = kL^{-1}\mathrm{d}L$ pc^{-3}, where the possible range of L lies between $0.1\,L_\odot$ and $1000\,L_\odot$ (where $L_\odot$ is the solar luminosity), and that an all-sky survey is being carried out with a flux limit of f_0.

(a) By first considering the number of stars of intrinsic luminosity L with apparent fluxes greater than f_0, derive an expression for the *integral number counts* $N(f > f_0)$, i.e. the total number of stars observed with fluxes greater than f_0.   [4]

(b) By considering the total luminosity of the stars detected in the survey, obtain a value for the average intrinsic luminosity of the survey stars, and compare this with the equivalent average intrinsic luminosity for a volume-limited survey. Comment on your results.   [6]

(c) Give two physical reasons why you might expect the integral number counts to increase less rapidly as f_0 is reduced than predicted by your result in part (a).   [2]

$\Big[$*You may assume that a source of luminosity, L, at distance d has an apparent flux, f, of* $\dfrac{L}{4\pi d^2}.\Big]$

2   Write brief notes on **two** of the following:

(a) the design features of radial velocity spectrometers for the detection of extra-solar planets;   [10]

(b) the properties of thermal Bremsstrahlung radiation and its use for diagnostic purposes in astrophysics;   [10]

(c) the rationale, principles and shortcomings of laser guide stars for adaptive optics;   [10]

(d) the trigonometric and spectroscopic parallax methods for distance estimation.   [10]

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3   Write an essay on the process of *image recovery from interferometer data* in astrophysics. Your essay should discuss the basic rationale and physics underlying this method of imaging, paying particular attention to how the parameters of the measurement such as telescope spacing, etc., affect the images recovered from interferometric data. Quantitative mention of typical scales and dimensions should be included where appropriate. [*You may assume that no atmospheric fluctuations are present when the interferometric data are secured.*]   [20]

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SECTION K   MEDICAL PHYSICS

1   What is the meaning of the term *therapeutic ratio* in radiotherapy?   [5]

Assume a cylindrical patient of diameter 20 cm, containing a centrally placed spherical planning target volume (PTV) of diameter 8 cm is being treated with 6 MV X-rays. Describe a suitable arrangement of (i) three fields and (ii) four fields to give a uniform distribution to the PTV.   [4]

Describe two methods of shaping the fields, and describe the advantages and disadvantages of each method.   [5]

A beam of 6 MV X-rays, at a source-skin distance of 100 cm, gives an output of 1.00 cGy per monitor unit at the depth of maximum dose (1.5 cm), and has a percentage depth dose of 73.8 % at a depth of 10 cm. Calculate how many monitor units per field will be required to give 2 Gy to the isocentre of an isocentric four-field plan, assuming a source-isocentre distance of 100 cm.   [6]

2   Write brief notes on **two** of the following:

(a) point matching, chamfer matching and mutual information for the registration of medical images;   [10]

(b) the variation with energy and atomic number of the mass attenuation coefficients for X-rays between 10 keV and 10 MeV (including a sketch of the variation of mass attenuation coefficient with energy for water and lead);   [10]

(c) acoustic impedance matching in the transmission of a diagnostic ultrasound pulse into the body (including a description of two related artefacts);   [10]

(d) absorbed dose, KERMA, equivalent dose and effective dose.   [10]

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3   (a) Limiting resolution is often expressed in terms of a number of line-pairs per mm. Describe what is meant by limiting resolution and the concept of a line-pair. If an X-ray film has a limiting resolution of $20\,\text{lp}\,\text{mm}^{-1}$, what is the size of the smallest object that can be resolved? [4]

(b) Explain why an intensifying screen is normally used in film radiography. Why does the introduction of an intensifying screen effect the resolution achievable? What limiting resolution is normally achieved by film-screen systems? [4]

(c) Briefly describe what is meant by the Modulation Transfer Function (MTF) in an imaging context, and describe its value in assessing an imaging system. Sketch the MTFs for a film imaging system and a film-screen imaging system. [6]

(d) In addition to the limiting resolution of the detector, the resolution achieved can also depend on the size of the source. Explain why this is the case. An object is placed $40\,\text{cm}$ in front of an X-ray film. The object is imaged using a source $1\,\text{mm}$ in length which is $120\,\text{cm}$ from the film. Would the sharpness of the final image be significantly degraded by the introduction of an intensifying screen? [6]

SECTION L   QUANTUM INFORMATION

1   The z component of the spin of a particle is measured by an ideal von Neumann measuring device. The Hamiltonians of the free particle and the pointer of the measuring device can be ignored during the measurement process so that the Hamiltonian describing the coupling of the particle to the pointer is

$$H = g(t)\lambda S_z P \,,$$

where $g(t)$ is 1 between $t = 0$ and $t = T$ and is 0 otherwise, λ is a coupling constant, S_z is the z spin operator for the particle, and P is the momentum operator of the pointer.

Find an expression for the time-evolution operator of the entire system for times $0 < t < T$ in terms of the eigenstates, $|a\rangle$, of the S_z operator. [5]

Determine the state, $|\Phi(T)\rangle$, of the system at time T given that the state of the system at time $t = 0$ is

$$|\Phi(0)\rangle = \sum_a \alpha_a |a\rangle \otimes |\psi(x)\rangle \,,$$

where $|\psi(x)\rangle$ is the initial state vector of the pointer which is a narrow wavepacket centred at $x = 0$, and α_a is the amplitude of spin eigenstate $|a\rangle$ of the particle. [5]

Comment on the difference between the states of the system at times $t = 0$ and $t = T$. [2]

Briefly discuss the extent to which this model describes the outcome of the measurement involving the particle and the measuring device according to:

    (a) the Copenhagen interpretation; [2]

    (b) the many-worlds interpretation; [2]

    (c) a local-hidden variables theory; and [2]

    (d) quantum state diffusion. [2]

2   Write brief notes on **two** of the following:

    (a) weak measurements; [10]

    (b) quantum teleportation; [10]

    (c) quantum cryptography. [10]

MN

3   A pendulum has mass m, angular frequency ω, and damping γ^{-1}. The pendulum is initially placed in a superposition of two states, one with the mass positioned at a horizontal distance x_1 from the vertical and the other at a distance x_2 with $x_1 > x_2$, where x_1 and x_2 are both very much smaller than the length of the pendulum. Outline the argument that leads to the following formula for the decoherence time,   [12]

$$t_{\mathrm{D}} = \frac{\hbar(x_1 + x_2)^2}{2\gamma m\omega(x_1^2 - x_2^2)^2}$$

.

Calculate the decoherence time for a pendulum of mass $100\,\mathrm{g}$ period $1\,\mathrm{s}$ and damping time $10^2\,\mathrm{s}$ when $x_1 - x_2$ is $10^{-7}\,\mathrm{m}$ and $x_1 - x_2 \ll x_1, x_2$.   [2]

What is the value of t_{D} if the energy transferred to the environment is distributed amongst vibrational modes with frequencies of the order of $10^6\omega$?   [4]

Briefly discuss the relevance of decoherence to Schrödinger's Cat paradox.   [2]

SECTION M   BIOLOGICAL PHYSICS

1   Write brief notes on **two** of the following:

(a) single-molecule experiments;   [10]

(b) cytoskeletal filaments;   [10]

(c) synthesis of ATP;   [10]

(d) bacterial chemotaxis.   [10]

2   Outline the derivation of the Cable equation that governs the evolution of
the membrane potential V and the ionic current j through the membrane in the
axon of a nerve cell:
$$\frac{\partial^2 V}{\partial x^2} = \frac{2}{\kappa a}\left(j + C\frac{\partial V}{\partial t}\right)\ ,$$
where a is the radius of the axon, κ is the ionic conductance within the axon, and
C is the membrane capacitance per unit area.   [4]

Suppose that the channels in the membrane have ohmic conductance, so that
the current through the membrane varies as $j = g(V - V_0)$, where g is a constant
conductance per unit area and V_0 is the resting potential. Show that if a small
excess charge is injected locally into the axon at position $x = 0$ at time $t = 0$, the
response is
$$V - V_0 = \exp\left(-\frac{t}{\tau}\right) t^{-1/2} \exp\left(-\frac{x^2 \tau}{4\lambda^2 t}\right)\ ,$$
and find expressions for the length scale λ and the time scale τ.   [4]

Describe how ion pumps maintain the membrane potential of axons at
$V_0 = -60$ mV, a voltage considerably more negative than the value that would
obtain at thermal equilibrium.   [4]

Describe the 'patch-clamp' technique, which has provided information about
the behaviour of individual sodium and potassium channels in the membrane.   [4]

Explain qualitatively how the observed response of the channels to changes
in the local membrane potential leads to the propagation of an action potential
along an axon.   [4]

MN

3   Describe briefly how the transcription of genes is regulated by transcription factors. [4]

How does a transcription factor rapidly locate its conjugate binding site on a DNA molecule? [4]

Protein X is a transcription factor which can exist in two different conformational states, labelled $\mathbf{r}$ and $\mathbf{t}$. In state $\mathbf{r}$, the factor binds to the operator region and represses transcription; in state $\mathbf{t}$, the factor does not bind to the operator and the gene is transcribed. X also binds a small ligand L. When the ligand is bound, the free energy of the state $\mathbf{r}$ is lower than that of the state $\mathbf{t}$ by ϵ, but when the ligand is not bound, the free energy of the state $\mathbf{r}$ is higher than that of the state $\mathbf{t}$ by ϵ.

The dissociation constants of the ligand for states $\mathbf{r}$ and $\mathbf{t}$ are respectively K_r and K_t. Explain why $\dfrac{K_\mathrm{r}}{K_\mathrm{t}} = \exp\left(-\dfrac{2\epsilon}{k_\mathrm{B}T}\right)$. [2]

If the ligand concentration is c, determine the following conditional probabilities in terms of ϵ, c, and $c_{1/2} \equiv \sqrt{K_\mathrm{r}K_\mathrm{t}}$.

$p(\mathbf{r}|\mathrm{L})$ is the probability that X is in state $\mathbf{r}$, given that the ligand is bound;
$p(\mathbf{r}|0)$ is the probability that X is in state $\mathbf{r}$, given that ligand is not bound;
$p(\mathrm{L}|\mathbf{r})$ is the probability that the ligand is bound, given that X is in state $\mathbf{r}$;
$p(\mathrm{L}|\mathbf{t})$ is the probability that the ligand is bound, given that X is in state $\mathbf{t}$. [4]

Hence show that the probability that X is in state $\mathbf{r}$ is:

$$p(\mathbf{r}) = \frac{c\exp\left(\frac{\epsilon}{k_\mathrm{B}T}\right) + c_{1/2}}{(c + c_{1/2})\left[1 + \exp\left(\frac{\epsilon}{k_\mathrm{B}T}\right)\right]} \ ,$$

and sketch $p(\mathbf{r})$ as a function of c. [4]

In bacteria, the amino acid tryptophan is manufactured using a set of enzymes which are the products of a single gene. This gene is regulated by a transcription factor which binds tryptophan. In the light of the above result, explain what function this serves. [2]

END OF PAPER